CLASSES OF FUNCTIONS WITH IMPROVED ESTIMATES IN APPROXIMATION BY THE MAX-PRODUCT BERNSTEIN OPERATOR

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Received 30 April 2010
Accepted 29 November 2010

In this paper, we find large classes of positive functions, others than those in [1], having even a Jackson-type estimate, $\omega_1(f; 1/n)$, in approximation by the nonlinear max-product Bernstein operator. The uniform estimate of the order $O[\omega_1(f; 1/n)^2 + \omega_1(f; 1/n)]$ is achieved, while near to the endpoints 0 and 1, the better pointwise estimate of the order $\omega_1(f, \sqrt{x(1-x)/n})$ is obtained. Finally, we prove that besides the preservation of quasi-convexity found in [1], the nonlinear max-product Bernstein operator preserves the quasi-concavity too.

Keywords: Max-product Bernstein approximation operator; Jackson-type estimate; pointwise estimate; polygonal line; Lipschitz function; quasi-concave function.

Mathematics Subject Classification 2010: 41A20, 41A36, 41A17, 41A25, 41A29

1. Introduction

For a function $f : [0, 1] \rightarrow \mathbb{R}_+$, the Bernstein approximation operator of max-product kind is given by the formula (see, e.g., [3, p. 326])

$$B_n^{(MF)}(f)(x) = \frac{\max_{k=0}^n p_{n,k}(x)f\left(\frac{k}{n}\right)}{\max_{k=0}^n p_{n,k}(x)},$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and $\max_{k=0}^n p_{n,k}(x) = \max_{k=0,\ldots,n} \left\{ p_{n,k}(x) \right\}$. Notice that the max-product Bernstein operator is obtained from the linear Bernstein polynomial written in the form $B_n(f)(x) = \frac{\sum_{k=0}^n p_{n,k}(x)f(k/n)}{\sum_{k=0}^n p_{n,k}(x)}$, surprisingly replacing the “sum” operator by the “maximum” operator.
As it was proved in [1, 2], \( B_n^{(M)}(f) \) is a nonlinear (more exactly sublinear) operator, well-defined for all \( x \in \mathbb{R} \), and a piecewise rational function on \( \mathbb{R} \). Also, in [1] it was proved that \( B_n^{(M)}(f) \) possesses some interesting approximation and shape preserving properties. For example, while in general the order of uniform approximation was found to be \( \omega_1(f; 1/\sqrt{n}) \), however, for some subclasses of functions including for example the class of concave functions and also a subclass of the convex functions, the order of approximation is essentially better, namely \( \omega_1(f; 1/n) \). In addition, in the same paper [1] it was proved that \( B_n^{(M)}(f) \) is continuous for any positive function \( f \), preserves the monotonicity and the quasi-convexity of \( f \).

The goal of the present paper is to extend the above mentioned results. Section 2 lists some known results which will be used in the next sections. In Sec. 3, we first point out by an example that the order of approximation by \( B_n^{(M)}(f) \), in general, cannot be better than \( \omega_1(f; 1/\sqrt{n}) \). Then, we prove that the order of approximation of monotone, continuous and strictly positive polygonal lines by \( B_n^{(M)}(f) \) is \( O(1/n) \), which is not happening in the case of approximation by the linear Bernstein polynomials \( B_n(f)(x) \) (see [1, Sec. 6]).

As a nice consequence of the results in Sec. 3, in Sec. 4 we present the main approximation results of the paper. Thus, for a continuous and strictly positive function \( f \), we obtain the order of uniform approximation by \( B_n^{(M)}(f)(x) \) (with the constant in \( O \) depending on \( f \))

\[
O\left\{ n \left[ \omega_1\left( f; \frac{1}{n} \right) \right]^2 + \omega_1\left( f; \frac{1}{n} \right) \right\},
\]

which for the classes of Lipschitz functions \( \text{Lip} \alpha \) with \( \alpha \in (1/2, 1] \), gives the approximation order \( 1/n^{2\alpha-1} \). Note that for \( \alpha \in (2/3, 1] \), this is essentially better than the general order \( O[\omega_1(f; 1/\sqrt{n})] = O[1/n^{\alpha/2}] \). Also, for \( f \in \text{Lip} \alpha \) with \( \alpha \in (2/3, 1] \), the above order of approximation is, in general, better than the order of approximation given by the linear Bernstein polynomials \( B_n(f)(x) \).

Also, at the end of Sec. 4, a mixed pointwise-uniform estimate, essentially better near to the endpoints 0 and 1, is obtained.

Finally, in Sec. 5 we prove that \( B_n^{(M)}(f) \) preserves the quasi-concavity of \( f \). Note that because \( B_n^{(M)}(f) \) is not linear, this is not a direct consequence of the preservation of quasi-convexity already proved in [1].

2. Auxiliary Results

In this section we list some known results which will be useful in proving the main results of the paper. First, we need some notations as follows:

For any \( k, j \in \{0, 1, \ldots, n\} \), let us consider the functions \( f_{k,n,j} : [\frac{j}{n+1}, \frac{j+1}{n+1}] \to \mathbb{R} \),

\[
f_{k,n,j}(x) = m_{k,n,j}(x)f\left( \frac{x}{n} \right),
\]

where \( m_{k,n,j}(x) = \frac{(j-x)^{k-j}}{\binom{k}{j}} \). It is easy to check that for any \( k \geq j \), \( f_{k,n,j} \) is nondecreasing and for any \( k \leq j \), \( f_{k,n,j} \) is nonincreasing.
We need the following known results.

**Lemma 2.1 ([1]).** Let \( k, j \in \{0, 1, 2, \ldots, n\} \) and \( x \in \left[ \frac{j}{n+1}, \frac{j+1}{n+1} \right] \). The following assertions hold:

(i) If \( j \leq k \leq k+1 \leq n \), then \( 1 \geq m_{k,n,j}(x) \geq m_{k+1,n,j}(x) \);
(ii) If \( 0 \leq k \leq k+1 \leq j \), then \( m_{k,n,j}(x) \leq m_{k+1,n,j}(x) \leq 1 \).

**Proof.** (i) See [1, proof of Lemma 3.2, Case 1].
(ii) See [1, proof of Lemma 3.2, Case 2]. \( \square \)

**Lemma 2.2 ([1, Relationship (4.17)].** Let \( x \in [0, 1] \) and let \( j \in \{0, 1, \ldots, n\} \) be such that \( x \in \left[ \frac{j}{n+1}, \frac{j+1}{n+1} \right] \). Then, one has

\[
B_n^{(M)}(f)(x) = \bigvee_{k=0}^n f_{k,n,j}(x).
\]

**Lemma 2.3 ([1, Lemma 4.6]).** Let \( f : [0, 1] \to [0, \infty) \) be a concave function. Then

\[
|B_n^{(M)}(f)(x) - f(x)| \leq 2\omega_1 \left( f; \frac{1}{n} \right), \quad \text{for all } x \in [0, 1].
\]

**Lemma 2.4 ([1, Theorem 5.5]).** If \( f : [0, 1] \to \mathbb{R}_+ \) is nondecreasing then for all \( n \in \mathbb{N}, n \geq 1 \), \( B_n^{(M)}(f) \) is nondecreasing.

**Lemma 2.5 ([1, Corollary 5.6]).** If \( f : [0, 1] \to \mathbb{R}_+ \) is nonincreasing then for all \( n \in \mathbb{N}, n \geq 1 \), \( B_n^{(M)}(f) \) is nonincreasing.

**Remark.** By Lemmas 2.4 and 2.5 and by the monotonicity properties of the functions \( f_{k,n,j} \) mentioned before Lemma 2.1, we get that for \( j \in \{0, 1, \ldots, n\} \) and \( x \in \left[ \frac{j}{n+1}, \frac{j+1}{n+1} \right] \), \( B_n^{(M)}(f)(x) = \bigvee_{k=0}^n f_{k,n,j}(x) \) for any nondecreasing function \( f \) and \( B_n^{(M)}(f)(x) = \bigvee_{k=0}^n f_{k,n,j}(x) \) for any nonincreasing function \( f \).

**Lemma 2.6 ([1, Lemma 3.4]).** One has

\[
\bigvee_{k=0}^n p_{n,k}(x) = p_{n,j}(x), \quad \text{for all } x \in \left[ \frac{j}{n+1}, \frac{j+1}{n+1} \right], \quad j = 0, 1, \ldots, n,
\]

where \( p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \).

**Definition 2.7.** Let \( f : [0, 1] \to \mathbb{R} \) be continuous on \([0, 1]\). The function \( f \) is called:

(i) quasi-convex if

\[
f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \quad \text{for all } x, y, \lambda \in [0, 1],
\]

(see, e.g., [3, page 4, (iv)]);
(ii) quasi-concave, if \(-f\) is quasi-convex.
Remark. By [4], the continuous function \( f \) is quasi-convex on \([0,1]\) equivalently means that there exists a point \( c \in [0,1] \) such that \( f \) is nonincreasing on \([0,c]\) and nondecreasing on \([c,1]\). From the above definition, we easily get that the continuous function \( f \) is quasi-concave on \([0,1]\), equivalently means that there exists a point \( c \in [0,1] \) such that \( f \) is nondecreasing on \([0,c]\) and nonincreasing on \([c,1]\).

3. Approximation of Monotone Polygonal Lines

In this section we deal with the estimates in approximation of polygonal lines by the max-product Bernstein operator. Besides their own importance, these results will be useful to obtain the main approximation results in Sec. 4.

By [1, Theorem 4.1], it was proved that for an arbitrary positive and continuous function on \([0,1]\), the order of uniform approximation by the nonlinear Bernstein operator of max-product kind is, in general, \( \omega_1(f,1/\sqrt{n}) \).

Firstly, below we show by an example that for the whole class of positive and continuous functions on \([0,1]\), this is the best possible order of uniform approximation. More precisely, in what follows we give an example of simple monotone continuous polygonal line \( f \), such that the order of approximation of \( f \) by the nonlinear Bernstein operator of max-product kind is exactly \( \omega_1(f,1/\sqrt{n}) \).

Example. Let us consider the function \( f : [0,1] \to [0,\infty) \), \( f(x) = 0 \) if \( x \in [0,1/2] \) and \( f(x) = x - 1/2 \) if \( x \in [1/2,1] \). Then \( B_n^{(M)}(f)(1/2) - f(1/2) = B_n^{(M)}(f)(1/2) \).

It is easy to check that \( 1/2 \in \left[ \frac{m_n}{n+1}, \frac{m_n + 1}{n+1} \right] \) for all \( n \in \mathbb{N} \), where \( n_0 = \lfloor n/2 \rfloor \). Then, since \( f \) is nondecreasing, we get (see the Remark after Lemma 2.5) \( B_n^{(M)}(f)(1/2) = \bigvee_{k=n_0}^n f_{k,n,n_0}(1/2) = \bigvee_{k=n_0}^n \left( \frac{k}{n} \right) f\left( \frac{k}{n} \right) \). Take \( k_n = n_0 + \lfloor \sqrt{n} \rfloor \). This implies

\[
\bigvee_{k=n_0}^n f_{k,n,j}(1/2) \geq f_{k_n,n,n_0}(1/2) = \frac{n}{k_n} \int \frac{k_n}{n} \frac{k_n}{n} \frac{1}{2} = \frac{n}{k_n} \left( \frac{n}{n_0} \right) \left( \frac{k_n}{n} - \frac{n_0 + 1}{n + 1} \right) \geq \frac{n}{n_0} \left( \frac{k_n}{n} \right) \left( \frac{n_0 + 1}{n + 1} \right) = \frac{n}{n_0} \left( \frac{n + 1}{n} \right) \left( \frac{n_0 + 1}{n + 1} \right) = \frac{n}{n_0} \left( \frac{\sqrt{n} - 1}{n + 1} \right).
\]


where for $n$ sufficiently large we have $\frac{\sqrt{n} - 1}{n + 1} > 0$. Let us denote $n_1 = n - n_0$. We get

$$\frac{n}{k_n} = \frac{(n - k_n + 1)(n - k_n + 2) \cdots (n - n_0)}{(n_0 + 1)(n_0 + 2) \cdots k_n}$$

$$= \frac{(n_1 - \sqrt{n} + 1)(n_1 - \sqrt{n} + 2) \cdots n_1}{(n_0 + 1)(n_0 + 2) \cdots (n_0 + \sqrt{n})} \geq \left( \frac{n_1 - \sqrt{n} + 1}{n_0 + \sqrt{n}} \right)^{\sqrt{n}}.$$  

Since $n_0 \leq \frac{n}{2}$ and $n_1 \geq \frac{n}{2}$, we obtain

$$\left( \frac{n_1 - \sqrt{n} + 1}{n_0 + \sqrt{n}} \right)^{\sqrt{n}} \geq \left( \frac{n}{2} - \frac{\sqrt{n}}{2} \right)^{\sqrt{n}}.$$  

Since $\lim_{n \to \infty} \left( \frac{n}{2} - \frac{\sqrt{n}}{2} \right)^{\sqrt{n}} = e^{-4}$, it follows that for sufficiently large $n$ we have $\frac{n}{k_n} \geq e^{-5}$. This implies

$$B_n^{(M)}(f)(1/2) \geq e^{-5} \cdot \frac{\sqrt{n} - 1}{n + 1} \geq \frac{e^{-5}}{6\sqrt{n}}$$

for sufficiently large $n$. Taking into account that $\omega_1(f, 1/\sqrt{n}) = \frac{1}{\sqrt{n}}$ for all $n \geq 4$, we get

$$B_n^{(M)}(f)(1/2) = B_n^{(M)}(f)(1/2) - f \left( \frac{1}{2} \right) \geq \frac{e^{-5}}{6} \omega_1(f, 1/\sqrt{n})$$

for sufficiently large $n$, which proves the desired conclusion.

However, as it was pointed out already in Introduction (see also Lemma 2.3), there exist subclasses of continuous functions such that the approximation order $\omega_1(f, 1/\sqrt{n})$, can be essentially improved to $\omega_1(f, 1/n)$.

In the same spirit of ideas, in the next part of this section we prove that for many types of continuous polygonal lines on the interval $[0, 1]$, we have the order of approximation $O(1/n) \equiv O(\omega_1(f, 1/n))$.

In the next Propositions 3.2–3.5 and in Theorem 3.7, all the functions will be assumed to be continuous and strictly positive on $[0, 1]$. In addition, although not explicitly mentioned in every proof, in all their proofs we may always assume that $B_n^{(M)}(f)(x) > f(x)$ and that $x \leq \frac{n}{n + 1}$.

This fact can be summarized by the following.

**Lemma 3.1.** Let $f : [0, 1] \to \mathbb{R}_+$.

(i) If at a point $x \in [0, 1]$ we have $B_n^{(M)}(f)(x) \leq f(x)$, then

$$|B_n^{(M)}(f)(x) - f(x)| \leq \omega_1 \left( f, \frac{1}{n} \right).$$
(ii) If \( x \in [\frac{n}{n+1}, 1] \) and \( f \) is nondecreasing on \([0, 1]\), then
\[
|B_n^{(M)}(f)(x) - f(x)| \leq \omega_1 \left( f, \frac{1}{n} \right).
\]

**Proof.** (i) Indeed, if \( B_n^{(M)}(f)(x) \leq f(x) \), then let \( j \in \{0, 1, \ldots, n\} \) be such that
\[
B_n^{(M)}(f)(x) = f(x) - B_n^{(M)}(f)(x) = f(x) - \sum_{k=0}^{n} f_{k,n,j}(x)
\]
\[
\leq f(x) - f_{j,n,j}(x) = f(x) - f\left( \frac{j}{n} \right)
\]
and since \( x, \frac{j}{n} \in [\frac{j}{n+1}, \frac{j+1}{n+1}] \), we get
\[
|B_n^{(M)}(f)(x) - f(x)| \leq \omega_1 \left( f, \frac{j+1}{n+1} \right) \leq \omega_1 \left( f, \frac{1}{n} \right).
\]

(ii) Now, if \( x \in [\frac{n}{n+1}, 1] \) and \( f \) is supposed to be nondecreasing, then by Lemma 2.4 it follows that \( B_n^{(M)}(f) \) is nondecreasing and noting that \( B_n^{(M)}(f)(1) = f(1) \), we get
\[
|B_n^{(M)}(f)(x) - f(x)| = B_n^{(M)}(f)(x) - f(x) \leq B_n^{(M)}(f)(1) - f(x)
\]
\[
= f(1) - f(x) \leq \omega_1 \left( f, \frac{1}{n+1} \right) \leq \omega_1 \left( f, \frac{1}{n} \right). \tag{\*}
\]

**Remark.** Notice that since \( B_n^{(M)}(f)(0) = f(0) = B_n^{(M)}(f)(1) - f(1) = 0 \), in all the approximation results we may assume that \( x \in (0, 1) \).

**Proposition 3.2.** Let us consider \( c \in [0, 1] \) and the continuous nondecreasing function \( f : [0, 1] \to [0, \infty) \), of the form
\[
f(x) = \begin{cases} 
1; & x \in [0, c], \\
ax + b; & x \in [c, 1],
\end{cases}
\]
that is, \( a \geq 0 \) and \( ac + b = 1 \). Then, for all \( n \in \mathbb{N} \) and all \( x \in [0, 1] \) we have the estimate
\[
|B_n^{(M)}(f)(x) - f(x)| \leq \frac{(a + 2)a}{n}.
\]

**Proof.** Firstly, note that from the Lemma 3.1, if \( B_n^{(M)}(f)(x) \leq f(x) \) or \( x > \frac{n}{n+1} \) then \( |B_n^{(M)}(f)(x) - f(x)| \leq \omega_1(f; 1/n) \leq \frac{a(a + 2)}{n} \) for all \( n \in \mathbb{N} \). Therefore, in what follows we can suppose that \( B_n^{(M)}(f)(x) > f(x) \) and \( x \leq \frac{n}{n+1} \).

Let \( x \in [0, 1] \) be fixed. We distinguish two cases: (i) \( x \in [c, 1] \) and (ii) \( x \in [0, c] \).
Case (i). Let \( j \in \{0, 1, \ldots, n-1\} \) be such that \( x \in \left[ \frac{j}{n+1}, \frac{j+1}{n+1} \right] \). (The case \( j = n \) can be excluded according to Lemma 3.1(ii)). Since \( f \) is nondecreasing we get \( B_n^M(f)(x) = \bigvee_{k=j}^n f_{k,n,j}(x) \).

Let us suppose that there exists \( k \in \{j+1, \ldots, n\} \) such that \( k \geq j+a \). Then, we have

\[
\frac{f_{k+1,n,j}(x)}{f_{k,n,j}(x)} = \frac{n-k}{k+1}, \frac{x}{1-x}, \frac{f(k+1/n)}{f(k/n)}.
\]

Since the function \( g(y) = \frac{y}{1-y} \) is nondecreasing on \( \left[ \frac{j}{n+1}, \frac{j+1}{n+1} \right] \), it follows that \( \frac{x}{1-x} \leq g\left(\frac{j+1}{n+1}\right) = \frac{j+1}{n-j} \) which combined with the fact that \( \frac{n}{n} \geq c \), gives us

\[
\frac{f_{k+1,n,j}(x)}{f_{k,n,j}(x)} \leq \frac{n-k}{k+1}, \frac{j+1}{n-j}, \frac{a}{n} + b
\]

Clearly, the function \( h(y) = \frac{ay+b+\frac{a}{n}}{ay+b} \) is nonincreasing and well defined on \([c, k/n]\). Indeed, by the continuity of \( f \) it follows that \( \frac{a}{n} + b \geq ac + b = 1 \). Since \( h \) is nonincreasing it follows that

\[
\frac{\frac{a}{n} + b + \frac{a}{n}}{\frac{a}{n} + b} = h\left(\frac{k}{n}\right) \leq h(c) = \frac{ac + b + \frac{a}{n}}{ac + b} = 1 + \frac{a}{n}
\]

\[
= \frac{n + a}{n} \leq \frac{j + 1 + a}{j + 1}.
\]

This implies

\[
\frac{f_{k+1,n,j}(x)}{f_{k,n,j}(x)} \leq \frac{n-k}{k+1}, \frac{j+1}{n-j}, \frac{j+1+a}{n+1}
\]

\[
\leq \frac{n-k}{n-j}, \frac{j+1+a}{k+1}.
\]

Since \( k \geq j+a \), it immediately follows that \( \frac{f_{k+1,n,j}(x)}{f_{k,n,j}(x)} \leq 1 \).

Therefore, for \( k \geq j+a \) we have \( f_{k+1,n,j}(x) \leq f_{k,n,j}(x) \) and since there exists \( k \in \{j+1, \ldots, n\} \) such that \( k \geq j+a \), then this implies \( B_n^M(f)(x) = \bigvee_{k \in J(a)} f_{k,n,j}(x) \) where \( J(a) = \{k \in \mathbb{N} : j \leq k \leq j+a\} \).

Note that if there does not exist \( k \in \{j+1, \ldots, n\} \) with \( k \geq j+a \), then \( J(a) = \{j, j+1, \ldots, n\} \).
Proposition 3.3. Note that the conclusion of the above proposition does not depend on Remark.

Case (ii). Where we have used the well-known inequality $\omega_{n,n,j}(x)$ and the proposition is proved.

Proof. If there exists $x_0 \in [0, c_2]$ such that $a_2 x_0 + b_2 = 1$, then we introduce the functions:

$$g(x) = \begin{cases} 1; & x \in [0, c_1], \\ a_1 x + b_1; & x \in [c_1, c_2], \\ a_2 x + b_2; & x \in [c_2, 1], \end{cases}$$

and the proposition is proved.
and

\[ h(x) = \begin{cases} 1; & x \in [0, x_0], \\ a_2 x + b_2; & x \in [x_0, 1]. \end{cases} \tag{3.1} \]

If \( a_2 x + b_2 > 1 \) for all \( x \in [0, 1] \), then we take \( h(x) = a_2 x + b_2 \) for all \( x \in [0, 1] \). We distinguish three cases: (i) \( x \in [c_2, 1] \); (ii) \( x \in [c_1, c_2] \) and (iii) \( x \in [0, c_1] \).

Case (i). Let \( j \in \{0, 1, \ldots, n - 1\} \) be such that \( x \in \left[ \frac{j}{n+1}, \frac{j+1}{n+1} \right] \). Then we have \( B_n^{(M)}(f)(x) = \mathcal{V}_{k,n,j} f_{k,n,j}(x) = \mathcal{V}_{k,n,j} m_{k,n,j}(x)f(h) \). Let \( k_0, \ldots, n+1 \) be such that \( B_n^{(M)}(f)(x) = f_{k_0,n,j}(x) \). If \( k_0 = j \) then \( k_0/n \leq j/(n+1) \) and \( B_n^{(M)}(f)(x) = f_{j,n,j}(x) = f(h) \) and it is immediate that

\[ B_n^{(M)}(f)(x) - f(x) = f(j/n) - f(x) \leq \omega_1(f, 1/n + 1) \]

\[ \leq \omega_1(f, 1/n) \leq \frac{\max \{ a_1, a_2 \}}{n}. \]

If \( k_0 > j \) then it is immediate that \( \frac{k_0}{n} \geq c_2 \), which implies \( f(k_0/n) = h(k_0/n) \) and therefore \( f_{k_0,n,j}(x) = h_{k_0,n,j}(x) \), where by definition

\[ h_{k_0,n,j}(x) = m_{k_0,n,j}(x)h(k/n). \]

We get \( B_n^{(M)}(f)(x) = f_{k_0,n,j}(x) = h_{k_0,n,j}(x) \leq B_n^{(M)}(h)(x) \) and because in this case \( f(x) = h(x) \) it follows that

\[ B_n^{(M)}(f)(x) - f(x) = B_n^{(M)}(f)(x) - h(x) \leq B_n^{(M)}(h)(x) - h(x). \]

If \( h \) is as in (3.1), then it satisfies the hypothesis of Proposition 3.2 and it follows that \( B_n^{(M)}(h)(x) - h(x) \leq \frac{(a_1 + 2)\omega_2}{n} \) which implies \( B_n^{(M)}(f)(x) - f(x) \leq \frac{(a_1 + 2)\omega_2}{n} \). If \( h \) is as in the second case, that is linear on \([0, 1]\), then \( h \) is a concave function and by Lemma 2.3 it follows that \( B_n^{(M)}(h)(x) - h(x) \leq 2\omega_1(f, 1/n) \leq \frac{2\omega_1}{n} \leq \frac{(a_2 + 2)\omega_2}{n} \) and again we get \( B_n^{(M)}(f)(x) - f(x) \leq \frac{(a_2 + 2)\omega_2}{n} \).

Case (ii). Let \( j \in \{0, 1, \ldots, n - 1\} \) be such that \( x \in \left[ \frac{j}{n+1}, \frac{j+1}{n+1} \right] \) and let \( k_0, \ldots, n+1 \) be such that \( B_n^{(M)}(f)(x) = f_{k_0,n,j}(x) \).

If \( k_0 = j \), then it is immediate that

\[ B_n^{(M)}(f)(x) - f(x) \leq \omega_1(f, 1/n) \leq \frac{\max \{ a_1, a_2 \}}{n}. \]

If \( \frac{k_0}{n} \in [c_1, c_2] \), then we get \( f_{k_0,n,j}(x) = g_{k_0,n,j}(x) \), where

\[ g_{k_0,n,j}(x) = m_{k_0,n,j}(x)g(k/n), \]

and this implies \( B_n^{(M)}(f)(x) = f_{k_0,n,j}(x) = g_{k_0,n,j}(x) \leq B_n^{(M)}(g)(x) \).
Since \( f(x) = g(x) \), it follows that
\[
B_n^{(M)}(f)(x) - f(x) = B_n^{(M)}(f)(x) - g(x) \leq B_n^{(M)}(g)(x) - g(x).
\]
Clearly, \( g \) satisfies the hypothesis of Proposition 3.2, which combined with the above inequality implies \( B_n^{(M)}(f)(x) - f(x) \leq \frac{(a_1+2)a_1}{n} \).

The last possibility is when \( \frac{k_0}{n} \in [c_2, 1] \). (Indeed, if we would have \( \frac{k_0}{n} < c_1 \) that would imply \( c_1 > \frac{k_0}{n} \geq \frac{k_0+1}{n} > \frac{k_0+1}{n+1} \geq x \), a contradiction with \( x \in [c_1, c_2] \)). Therefore, this implies \( f_{k_0,n,j}(x) = h_{k_0,n,j}(x) \), where it does not matter which \( h \) we choose. We have here two subcases: (ii) \( a_1 \geq a_2 \) and (ii) \( a_1 < a_2 \).

**Subcase (ii)**. By simple geometrical reasonings, it is immediate that \( f(\frac{k_0}{n}) = h(\frac{k_0}{n}) \), which immediately implies \( f_{k_0,n,j}(x) = h_{k_0,n,j}(x) \leq g_{k_0,n,j}(x) \) and further one, \( B_n^{(M)}(f)(x) \leq B_n^{(M)}(g)(x) \). This leads to the same conclusion as above, that is
\[
B_n^{(M)}(f)(x) - f(x) \leq B_n^{(M)}(g)(x) - g(x) \leq \frac{(a_1+2)a_1}{n}.
\]

**Subcase (ii)**. In this case, by simple geometrical reasonings we have \( f(x) \geq h(x) \) for all \( x \in [0, 1] \) (it does not matter here which definition for \( h \) we choose) and we get
\[
B_n^{(M)}(f)(x) - f(x) = h_{k_0,n,j}(x) - f(x) \leq h_{k_0,n,j}(x) - h(x) \leq B_n^{(M)}(h)(x) - h(x) .
\]

Clearly, in both definitions, \( h \) satisfies the hypothesis of Proposition 3.2, which combined with the above inequality implies \( B_n^{(M)}(f)(x) - f(x) \leq \frac{(a_2+2)a_2}{n} \).

**Case (iii)**. As in the proof of Proposition 3.2, we get \( B_n^{(M)}(f)(x) - f(x) \leq B_n^{(M)}(f)(c_1) - f(c_1) \) and since for \( c_1 \) the case (ii) is applicable we immediately obtain \( B_n^{(M)}(f)(x) - f(x) \leq \frac{\max \{1, 2\} (a_1+2)a_1}{n} \).

Collecting all the estimates in the above cases and subcases we get the desired conclusion.\( \square \)

**Proposition 3.4.** Let us consider the nondecreasing continuous function \( f : [0, 1] \rightarrow [0, \infty) \),
\[
f(x) = \begin{cases} 
\alpha; & x \in [0, c], \\
ax + b; & x \in [c, 1],
\end{cases}
\]
where \( \alpha > 0 \). Then, we have the estimate
\[
|B_n^{(M)}(f)(x) - f(x)| \leq \frac{(2 + \frac{a}{\alpha})a}{n}.
\]

**Proof.** Let us consider the function
\[
g(x) = \begin{cases} 
1; & x \in [0, c], \\
\frac{1}{\alpha}(ax + b); & x \in [c, 1].
\end{cases}
\]
By Proposition 3.2, we get \( |B_n(M)(g)(x) - g(x)| \leq \frac{(a/\alpha + 2)a/\alpha}{n} \). By the homogeneity of \( B_n(M) \), we get
\[
|B_n(M)(f)(x) - f(x)| = |B_n(M)(\alpha g)(x) - \alpha g(x)| = \alpha|B_n(M)(g)(x) - g(x)|.
\]
This implies the desired conclusion.

**Proposition 3.5.** Let us consider the nondecreasing continuous function \( f: [0, 1] \to [0, \infty) \),
\[
f(x) = \begin{cases} 
\alpha; & x \in [0, c_1], \\
ax + b_1; & x \in [c_1, c_2], \\
ax + b_2; & x \in [c_2, 1],
\end{cases}
\]
where \( \alpha > 0 \). Then, we have the estimate
\[
|B_n(M)(f)(x) - f(x)| \leq \left( \frac{\max_{i \in \{1, 2\}} \left( 2 + \frac{a_i}{\alpha} \right)a_i}{n} \right).
\]

**Proof.** The proof is analogous with the proof of Proposition 3.4, and so we omit it.

**Remark.** By Propositions 3.4 and 3.5 it follows that if \( f \) is a strictly positive function on \([0, 1]\) and satisfies the hypothesis in Proposition 3.4 or Proposition 3.5, then we have
\[
|B_n(M)(f)(x) - f(x)| \leq \left( \frac{2 + \frac{\alpha_0}{f(0)}}{n} \right) \alpha_0, \quad x \in [0, 1],
\]
where in the first case we have \( \alpha_0 = a \) and in the second case we have \( \alpha_0 = \max\{a_1, a_2\} \).

In what follows, we extend the above results to any monotone, continuous and strictly positive polygonal line on \([0, 1]\).

**Definition 3.6.** Let \( a, b \in \mathbb{R} \), \( a < b \) and let \( a = x_0 < x_1 < \cdots < x_l = b \) be a division of the interval \([a, b]\). A function \( f: [a, b] \to \mathbb{R} \), will be called a continuous polygonal line if \( f \) is continuous on \([a, b]\) and for any \( i \in \{0, 1, \ldots, l - 1\} \), there exists a polynomial function of degree less then or equal to 1, \( f_i: \mathbb{R} \to \mathbb{R} \), such that \( f(x) = f_i(x) = ax + b_i \) for all \( x \in [x_i, x_{i+1}] \). We denote \( f = (f_0[x_0, x_1], f_1[x_1, x_2], \ldots, f_{l-1}[x_{l-1}, x_l]) \).

**Theorem 3.7.** Let \( f = (f_0[x_0, x_1], f_1[x_1, x_2], \ldots, f_{l-1}[x_{l-1}, x_l]) \), \( f_i(x) = ax_i + b_i \), be a continuous, nondecreasing and strictly positive on \([0, 1]\) polygonal line. Then, for all \( x \in [0, 1] \) and \( n \in \mathbb{N} \), we have the estimate
\[
|B_n(M)(f)(x) - f(x)| \leq \left( \frac{2 + \frac{a_{i_0}}{f(0)}}{n} \right) a_{i_0},
\]
where \( a_{i_0} = \max\{a_0, a_1, \ldots, a_{l-1}\} \).
Proof. We prove the theorem by mathematical induction on the variable $l \in \{1, 2, \ldots, \}$, representing the number of intervals given by the division of the interval $[0, 1]$.

If $l = 1$ then it is immediate that $f$ is linear of the form $f(x) = ax + b$, $x \in [0, 1]$. Then, by Lemma 2.3 it follows that

$$|B_{n}(f)(x) - f(x)| \leq 2\omega_{1}\left(\frac{1}{n}\right) \leq \frac{2a}{n} \leq \frac{(2 + \frac{a}{f(0)})}{n}.$$  

Suppose now that the assertion of the theorem holds for $l - 1$. We denote by $\overline{a} = \max\{a_{0}, a_{1}, \ldots, a_{l-2}\}$. Also, we need the functions

$$g = (f_{0}[x_{0}, x_{1}], f_{1}[x_{1}, x_{2}], \ldots, f_{l-2}[x_{l-2}, x_{l}])$$

and

$$h(x) = \begin{cases} f(0); & x \in [0, c], \\ a_{0}x + f(x_{l-1}) - a_{0}x_{l-1}; & x \in [c, x_{l-1}], \\ f(x) = f_{l-1}(x); & x \in [x_{l-1}, 1], \end{cases}$$

where $c \in [0, 1]$ is such that $a_{0}c + f(x_{l-1}) - a_{0}x_{l-1} = f(0)$. Since $a_{0} = \max\{a_{0}, a_{1}, \ldots, a_{l-1}\}$, by simple geometrical reasonings we get $f(x) \geq h(x)$ for all $x \in [0, 1]$. In addition, it is easy to check that $h$ is continuous on $[0, 1]$.

For arbitrary $x \in [0, 1]$, we distinguish two cases: (i) $x \in [0, x_{l-1}]$ and (ii) $x \in [x_{l-1}, 1]$.

Case (i). Let $j \in \{0, 1, \ldots, n - 1\}$ be such that $x \in \left[\frac{j}{n} + \frac{1}{n+1}, \frac{j+1}{n+1}\right]$ and let $k_{0} \in \{j, \ldots, n\}$ be such that $B_{n}(f)(x) = f_{k_{0}, n, j}(x)$. If $\frac{k_{0}}{n} \leq x_{l-1}$, then it is immediate that $f_{k_{0}, n, j}(x) = g_{k_{0}, n, j}(x)$ which immediately implies $B_{n}(f)(x) \leq B_{n}(g)(x)$. Recall here that everywhere in the proof we denoted $f_{k, n, j}(x) = m_{k, n, j}(x)f(k/n)$, $g_{k, n, j}(x) = m_{k, n, j}(x)g(k/n)$ and $h_{k, n, j}(x) = m_{k, n, j}(x)h(k/n)$.

Since $g$ is split in $l - 1$ intervals, from our assumption we get

$$|B_{n}(g)(x) - g(x)| \leq \frac{(2 + \frac{\overline{a}}{g(0)})}{n}\overline{a}.$$  

Since $g(0) = f(0)$ and $\overline{a} \leq a_{0}$, we get

$$B_{n}(f)(x) - f(x) = B_{n}(g)(x) - g(x) \leq B_{n}(g)(x) - g(x) \leq \frac{(2 + \frac{\overline{a}}{g(0)})}{n}\overline{a} \leq \frac{(2 + \frac{a_{0}}{f(0)})}{n}a_{0}.$$
By direct calculation we get
\[ g \]
where
\[ \text{Lemma 3.8. For any function } f : [0, 1] \to [0, \infty), \text{ we have } \]
\[ B_n^{(M)}(f)(x) = B_n^{(M)}(f)(1 - x), \quad x \in [0, 1], \]
where \( g(x) = f(1 - x) \) for all \( x \in [0, 1] \).

**Proof.** By direct calculation we get
\[ B_n^{(M)}(g)(1 - x) = \sum_{k=0}^{n} p_n,k(1 - x) g \left( \frac{k}{n} \right) = \sum_{k=0}^{n} p_n,k(1 - x) f \left( \frac{n - k}{n} \right) \]
\[ = B_n^{(M)}(f)(x). \]
Theorem 3.9. Let \( f = (f_0[x_0,x_1], f_1[x_1,x_2], \ldots, f_{l-1}[x_{l-1},x_l]), \) \( f_i(x) = a_i + b_i \), be a continuous, nonincreasing and strictly positive on \([0,1]\) polygonal line. Then, we have the estimate

\[
|B_n^{(M)}(f)(x) - f(x)| \leq \frac{\left(2 + \frac{\alpha_{i_0}}{f(1)}\right)\overline{\mu}}{n}, \quad x \in [0,1],
\]

where \( \overline{\mu} = \max\{|a_0|, |a_1|, \ldots, |a_{l-1}|\} \).

Proof. Consider the function \( g: [0,1] \to [0,\infty), \) \( g(x) = f(1-x) \). Then, evidently \( g \) is nondecreasing and \( g \) has the form

\[
g = (g_0[y_0,y_1], g_1[y_1,y_2], \ldots, g_{l-1}[y_{l-1},y_l]),
\]

where \( y_i = 1 - x_{i-1}, i \in \{0,1,\ldots,l\} \) and \( g_i(x) = f_{i-1}(1-x) = c_i x + d_i, i \in \{0,1,\ldots,l-1\} \). Moreover, it is easy to check that

\[
\max\{c_0, c_1, \ldots, c_{l-1}\} = \max\{|a_0|, |a_1|, \ldots, |a_{l-1}|\} =: \overline{\mu}.
\]

By Theorem 3.7 it follows that

\[
|B_n^{(M)}(g)(x) - g(x)| \leq \frac{\left(2 + \frac{\alpha_{i_0}}{g(0)}\right)\overline{\mu}}{n}.
\]

Taking into account the above Lemma 3.8, we obtain

\[
|B_n^{(M)}(f)(x) - f(x)| = |B_n^{(M)}(g)(1-x) - g(1-x)| \leq \frac{\left(2 + \frac{\alpha_{i_0}}{g(0)}\right)\overline{\mu}}{n}
\]

\[
= \frac{\left(2 + \frac{\alpha_{i_0}}{f(1)}\right)\overline{\mu}}{n}
\]

and the theorem is proved. \( \Box \)

4. Main Approximation Results

This section contains the main approximation results of the paper. They are obtained as consequences of the results on the approximation of polygonal lines in Sec. 3. As in the previous section, in all the proofs of the approximation results, according to Lemma 3.1 we may always assume that \( B_n^{(M)}(f(x)) > f(x) \).

Theorem 4.1. If \( f: [0,1] \to [0,\infty) \) is a continuous, nondecreasing and strictly positive function on \([0,1]\), then we have the estimate

\[
|B_n^{(M)}(f)(x) - f(x)| \leq \left(\frac{n \omega_1\left(f, \frac{1}{n}\right)}{f(0)} + 3\right) \omega_1\left(f, \frac{1}{n}\right), \quad x \in [0,1], \quad n \in \mathbb{N}.
\]
Proof. For $n \in \mathbb{N}$, we consider the function
\[ g = \{g_0[x_0, x_1], g_1[x_1, x_2], \ldots, g_{n-1}[x_{n-1}, x_n]\}, \]
where $x_i = \frac{i}{n}$, $i \in \{0, 1, \ldots, n\}$ and $g_{i-1}(x) = \frac{(x-x_i)(f(x_i)-f(x_{i-1}))}{x_i-x_{i-1}} + f(x_i)$ for all $x \in [x_{i-1}, x_i]$, $i \in \{1, \ldots, n\}$. Since \( f \left( \frac{k}{n} \right) = g \left( \frac{k}{n} \right) \) for all \( k \in \{0, 1, \ldots, n\} \), by the definition of $B_n^{(M)}(f)$ too, it follows that $B_n^{(M)}(f)(x) = B_n^{(M)}(g)(x)$ for all $x \in [0, 1]$. Also, it is immediate that $f(0) = g(0)$ and that $|f(x) - g(x)| \leq \omega_1(f, \frac{1}{n})$ for all $x \in [0, 1]$. Indeed, for $x \in [0, 1]$ let $i \in \{0, 1, \ldots, n-1\}$ be such that $x \in [x_i, x_{i+1}]$. Since $g$ is nondecreasing, by Theorem 3.7 we get
\[ |B_n^{(M)}(g)(x) - g(x)| \leq \frac{2 + a_{i_n}}{n} \frac{a_{i_n}}{g(0)}, \quad x \in [0, 1], \]
where
\[ a_{i_n} = \max_{i \in \{1, \ldots, n\}} \left( \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right). \]

We have $\frac{a_{i_n}}{n} = f(x_{i_n+1}) - f(x_{i_n}) \leq \omega_1(f, \frac{1}{n})$. On the other hand, it is immediate that $\frac{(x_{i_n}) - (x_{i+n})}{x_{i+n} - x_{i+n-1}} \leq n\omega_1(f, \frac{1}{n})$ for all $i \in \{1, \ldots, n\}$. Therefore, we obtain
\[ |B_n^{(M)}(g)(x) - g(x)| \leq \frac{n\omega_1(f, \frac{1}{n})}{f(0) + 2} \omega_1(f, \frac{1}{n}), \quad x \in [0, 1]. \]

For $x \in [0, 1]$, we get
\[ |B_n^{(M)}(f)(x) - f(x)| = |B_n^{(M)}(g)(x) - f(x)| \leq |B_n^{(M)}(g)(x) - g(x)| + |f(x) - g(x)| \]
\[ \leq \left( \frac{n\omega_1(f, \frac{1}{n})}{f(0) + 2} \right) \omega_1(f, \frac{1}{n}) + \omega_1(f, \frac{1}{n}) \]
\[ = \left( \frac{n\omega_1(f, \frac{1}{n})}{f(0) + 3} \right) \omega_1(f, \frac{1}{n}) \]
and the theorem is proved.
Corollary 4.2. If \( f : [0, 1] \to [0, \infty) \) is a continuous, nonincreasing and strictly positive function on \([0, 1]\), then we have the estimate

\[
|B_n^{(M)}(f)(x) - f(x)| \leq \left( \frac{n \omega_1 \left( f, \frac{1}{n} \right)}{f(1)} + 3 \right) \omega_1 \left( f, \frac{1}{n} \right), \quad x \in [0, 1], \quad n \in \mathbb{N}.
\]

Proof. Take \( g(x) = f(1 - x), \quad x \in [0, 1]. \) Clearly, \( g \) satisfies the hypothesis in Theorem 4.1, which means that

\[
|B_n^{(M)}(g)(x) - g(x)| \leq \left( \frac{n \omega_1 \left( g, \frac{1}{n} \right)}{g(0)} + 3 \right) \omega_1 \left( g, \frac{1}{n} \right), \quad x \in [0, 1].
\]

Since \( f(1) = g(0) \) and \( \omega_1(f, \frac{1}{n}) = \omega_1(g, \frac{1}{n}) \), by Lemma 3.8 too, for \( x \in [0, 1] \) we get

\[
|B_n^{(M)}(f)(x) - f(x)| = |B_n^{(M)}(g)(1 - x) - g(1 - x)|
\]

\[
\leq \left( \frac{n \omega_1 \left( g, \frac{1}{n} \right)}{g(0)} + 3 \right) \omega_1 \left( g, \frac{1}{n} \right)
\]

\[
= \left( \frac{n \omega_1 \left( f, \frac{1}{n} \right)}{f(1)} + 3 \right) \omega_1 \left( f, \frac{1}{n} \right),
\]

which proves the corollary.

In all that follows, for a continuous function \( f : [0, 1] \to \mathbb{R} \), we denote \( m_f = \min \{ f(x) : x \in [0, 1] \} \).

Theorem 4.3. If \( f : [0, 1] \to [0, \infty) \) is a continuous, quasi-convex and strictly positive function on \([0, 1]\), then we have the estimate

\[
|B_n^{(M)}(f)(x) - f(x)| \leq \left( \frac{n \omega_1 \left( f, \frac{1}{n} \right)}{m_f} + 3 \right) \omega_1 \left( f, \frac{1}{n} \right), \quad x \in [0, 1], \quad n \in \mathbb{N}.
\]

Proof. Since \( f \) is quasi-convex, it follows that there exists \( c \in [0, 1] \) such that \( f \) is nonincreasing on \([0, c]\) and nondecreasing on \([c, 1]\). In addition, it is immediate that \( f(c) = m_f \). Let us introduce the functions

\[
g(x) = \begin{cases} 
m_f; & x \in [0, c], 
\end{cases}
\]

\[
g(x) = \begin{cases} 
f(x); & x \in [c, 1].
\end{cases}
\]
In addition, we observe that the hypothesis in Corollary 4.2. Therefore, we have
\[
\max \{\omega (g, \frac{1}{n}), \omega (h, \frac{1}{n})\} \leq \omega (f, \frac{1}{n}).
\]
Since \( f = g \vee h \), by the relation (2.7) in [1] satisfied by \( B_n^{(M)} \), we can write
\[
B_n^{(M)}(f)(x) = B_n^{(M)}(g)(x) \vee B_n^{(M)}(h)(x), \quad x \in [0,1].
\]
In addition, we observe that \( g \) satisfies the hypothesis in Theorem 4.1 and \( h \) satisfies the hypothesis in Corollary 4.2. Therefore, we have
\[
|B_n^{(M)}(g)(x) - g(x)| \leq \left( \frac{n \omega_1 \left( g, \frac{1}{n} \right)}{g(0)} + 3 \right) \omega_1 \left( g, \frac{1}{n} \right), \quad x \in [0,1]
\]
and
\[
|B_n^{(M)}(h)(x) - h(x)| \leq \left( \frac{n \omega_1 \left( h, \frac{1}{n} \right)}{h(1)} + 3 \right) \omega_1 \left( h, \frac{1}{n} \right), \quad x \in [0,1].
\]
Let us choose arbitrary \( x \in [0,1] \). If \( B_n^{(M)}(f)(x) = B_n^{(M)}(g)(x) \), then we have
\[
B_n^{(M)}(f)(x) - f(x) = B_n^{(M)}(g)(x) - g(x) \vee h(x) \leq B_n^{(M)}(g)(x) - g(x)
\]
\[
\leq \left( \frac{n \omega_1 \left( g, \frac{1}{n} \right)}{m_f} + 3 \right) \omega_1 \left( g, \frac{1}{n} \right)
\]
\[
\leq \left( \frac{n \omega_1 \left( f, \frac{1}{n} \right)}{m_f} + 3 \right) \omega_1 \left( f, \frac{1}{n} \right).
\]
If \( B_n^{(M)}(f)(x) = B_n^{(M)}(h)(x) \), then we have
\[
B_n^{(M)}(f)(x) - f(x) = B_n^{(M)}(h)(x) - g(x) \vee h(x) \leq B_n^{(M)}(h)(x) - h(x)
\]
\[
\leq \left( \frac{n \omega_1 \left( h, \frac{1}{n} \right)}{m_f} + 3 \right) \omega_1 \left( h, \frac{1}{n} \right)
\]
\[
\leq \left( \frac{n \omega_1 \left( f, \frac{1}{n} \right)}{m_f} + 3 \right) \omega_1 \left( f, \frac{1}{n} \right).
\]
This proves the theorem.
Theorem 4.4. If \( f : [0, 1] \to [0, \infty) \) is a continuous, quasi-concave and strictly positive function on \([0, 1]\), then we have the estimate

\[
|B_n^{(M)}(f)(x) - f(x)| \leq \left( \frac{n \omega_1\left( f, \frac{1}{n} \right)}{m_f} + 3 \right) \omega_1\left( f, \frac{1}{n} \right), \quad x \in [0, 1], \quad n \in \mathbb{N}.
\]

Proof. Since \( f \) is quasi-concave, it follows that there exists \( c \in [0, 1] \) such that \( f \) is nondecreasing on \([0, c]\) and nonincreasing on \([c, 1]\). Let us introduce the functions

\[
g(x) = \begin{cases} f(x); & x \in [0, c], \\ f(c); & x \in [c, 1]. \end{cases}
\]

and

\[
h(x) = \begin{cases} f(c); & x \in [0, c], \\ f(x); & x \in [c, 1]. \end{cases}
\]

It is immediate that \( f(0) = g(0), f(1) = h(1) \) and that \( \max\{\omega_1(g, \frac{1}{n}), \omega_1(h, \frac{1}{n})\} \leq \omega_1(f, \frac{1}{n}) \). In addition, since \( f \leq g \) and \( f \leq h \), by the monotonicity of \( B_n^{(M)} \), we get

\[
B_n^{(M)}(f)(x) \leq \min\{B_n^{(M)}(g)(x), B_n^{(M)}(h)(x)\}, \quad x \in [0, 1].
\]

In order to prove our assertion, we distinguish two cases: (i) \( x \in [0, c] \) and (ii) \( x \in [c, 1] \).

Case (i). Noting that \( f(x) = g(x) \) and that \( g \) satisfies the hypothesis in Theorem 4.1, we get

\[
B_n^{(M)}(f)(x) - f(x) = B_n^{(M)}(f)(x) - g(x) \leq B_n^{(M)}(g)(x) - g(x)
\]

\[
\leq \left( \frac{n \omega_1\left( g, \frac{1}{n} \right)}{g(0)} + 3 \right) \omega_1\left( g, \frac{1}{n} \right)
\]

\[
\leq \left( \frac{n \omega_1\left( f, \frac{1}{n} \right)}{f(0)} + 3 \right) \omega_1\left( f, \frac{1}{n} \right).
\]

Case (ii). Noting that \( f(x) = h(x) \) and that \( h \) satisfies the hypothesis in Corollary 4.2, we get

\[
B_n^{(M)}(f)(x) - f(x) = B_n^{(M)}(f)(x) - h(x) \leq B_n^{(M)}(h)(x) - h(x)
\]

\[
\leq \left( \frac{n \omega_1\left( h, \frac{1}{n} \right)}{h(1)} + 3 \right) \omega_1\left( h, \frac{1}{n} \right).
\]
\[ \leq \left( \frac{n\omega_1 \left( f, \frac{1}{n} \right)}{f(1)} + 3 \right) \omega_1 \left( f, \frac{1}{n} \right). \]

Collecting all the estimates in the above cases (i) and (ii) and since \( m_f = \min\{f(0), f(1)\} \), we easily get the estimate in the statement.

**Theorem 4.5.** Let \( f : [0, 1] \to [0, \infty) \) be a continuous and strictly positive function and suppose that there exists a division of the interval \([0, 1]\), \( 0 = x_0 < x_1 < \cdots < x_l = 1 \) such that \( f \) is monotone on each interval \([x_i, x_{i+1}]\), \( i \in \{0, 1, \ldots, l - 1\} \) and of opposite monotonicity on each two consecutive intervals. Then

\[ |B_n(M)(f)(x) - f(x)| \leq \left( \frac{n\omega_1 \left( f, \frac{1}{n} \right)}{m_f} + 3 \right) \omega_1 \left( f, \frac{1}{n} \right), \quad x \in [0, 1], \quad n \in \mathbb{N}, \]

where \( m_f = \min\{f(x) ; x \in [0, 1]\} \).

**Proof.** We prove the theorem by mathematical induction on the variable \( l \in \{1, 2, \ldots\} \) representing the number of intervals given by the division of the interval \([0, 1]\). If \( l = 1 \) then it is immediate that \( f \) is monotone and the conclusion follows from Theorem 4.1 or Corollary 4.2, respectively. If \( l = 2 \) then the conclusion follows from Theorem 4.3 or Theorem 4.4, respectively. Suppose now that the conclusion of the lemma holds for any \( p, 1 \leq p \leq l - 1 \). We have two cases: (i) \( f \) is nonincreasing on \([x_{l-1}, 1]\) and (ii) \( f \) is nondecreasing on \([x_{l-1}, 1]\).

Case (i). First, we define the function \( g(x) = \begin{cases} f(x); & x \in [0, x_{l-1}], \\ f(x_{l-1}); & x \in [x_{l-1}, 1]. \end{cases} \)

Then, we introduce the function \( h \) depending on the value \( f(x_{l-1}) \). If \( x_{l-1} \) is the global maximum point of \( f \) then we consider

\[ h(x) = \begin{cases} f(x_{l-1}); & x \in [0, x_{l-1}], \\ f(x); & x \in [x_{l-1}, 1]. \end{cases} \]

Otherwise, let \( c \in [0, x_{l-2}] \) be the point of maximum value where the graph of \( f \) intersects the line \( y = f(x_{l-1}) \). We define

\[ h(x) = \begin{cases} f(x); & x \in [0, c], \\ f(x_{l-1}); & x \in [c, x_{l-1}], \\ f(x); & x \in [x_{l-1}, 1]. \end{cases} \]
Since on the interval \([x_{l-2}, 1]\), \(g\) is monotone, it follows that the interval \([0, 1]\) can be split in \(p\) intervals, \(p < l\), satisfying the hypothesis in the present theorem. This statement holds for \(h\) too. From our assumption it follows that

\[
|B_n^M(g)(x) - g(x)| \leq \left(\frac{n\omega_1(\frac{g, 1}{n})}{m_g} + 3\right) \omega_1\left(\frac{g, 1}{n}\right), \quad x \in [0, 1]
\]

and

\[
|B_n^M(h)(x) - h(x)| \leq \left(\frac{n\omega_1(\frac{h, 1}{n})}{m_h} + 3\right) \omega_1\left(\frac{h, 1}{n}\right), \quad x \in [0, 1].
\]

Now, let us choose arbitrary \(x \in [0, 1]\). If \(x \in [0, x_{l-1}]\) then \(f(x) = g(x)\) and \(B_n^M(f)(x) \leq B_n^M(g)(x)\). This implies

\[
B_n^M(f)(x) - f(x) \leq \left(\frac{n\omega_1(\frac{g, 1}{n})}{m_g} + 3\right) \omega_1\left(\frac{g, 1}{n}\right).
\]

It is easy to check that \(\omega_1(\frac{g, 1}{n}) \leq \omega_1(\frac{f, 1}{n})\) and that \(m_f \leq m_g\). Therefore, we obtain the desired conclusion in this case.

If \(x \in [x_{l-1}, 1]\) then \(f(x) = h(x)\) and \(B_n^M(f)(x) \leq B_n^M(h)(x)\). This implies

\[
B_n^M(f)(x) - f(x) \leq \left(\frac{n\omega_1(\frac{h, 1}{n})}{m_h} + 3\right) \omega_1\left(\frac{h, 1}{n}\right).
\]

Again, it is easy to prove that \(\omega_1(\frac{h, 1}{n}) \leq \omega_1(\frac{f, 1}{n})\) and that \(m_f \leq m_h\). Hence, we get the conclusion of the theorem in this case too.

**Case (ii).** We construct the function \(g\) exactly as in the above case (i). If \(x_{l-1}\) is a global minimum point for \(f\), then we take

\[
h(x) = \begin{cases} 
  f(x_{l-1}); & x \in [0, x_{l-1}], \\
  f(x); & x \in [x_{l-1}, 1].
\end{cases}
\]

Otherwise, let \(c \in [0, x_{l-2}]\) be the point of maximum value where the graph of \(f\) intersects the line \(y = f(x_{l-1})\). We take

\[
h(x) = \begin{cases} 
  f(x); & x \in [0, c], \\
  f(x_{l-1}); & x \in [c, x_{l-1}], \\
  f(x); & x \in [x_{l-1}, 1].
\end{cases}
\]
Clearly, we may suppose that for $g$ and $h$ we have the same estimations as in the above case (i). Since $f = g \sqrt{h}$ it follows that

$$B^{(M)}_n(f)(x) = B^{(M)}_n(g)(x) \sqrt{B^{(M)}_n(h)(x)}.$$ 

From now on the proof goes on the same pattern as in the proof of Theorem 4.3 and noting that $\max\{\omega_1(g, \frac{1}{n}), \omega_1(h, \frac{1}{n})\} \leq \omega_1(f, \frac{1}{n})$ and that $m_f = \min\{m_g, m_h\}$ we easily get the desired conclusion in this case too and the proof is complete. \(\square\)

We present now the following more general result.

**Theorem 4.6.** Let $f: [0, 1] \to [0, \infty)$ be a continuous and strictly positive function. Then

$$|B^{(M)}_n(f)(x) - f(x)| \leq \left( \frac{\omega_1(f, \frac{1}{n})}{m_f} + 4 \right) \omega_1(f, \frac{1}{n}), \quad x \in [0, 1], \quad n \in \mathbb{N}, \quad (4.1)$$

where $m_f = \min\{f(x); x \in [0, 1]\}$.

**Proof.** As in the proof of Theorem 4.1, for $n \in \mathbb{N}$, we consider the function $g = (g_0[x_0, x_1], g_1[x_1, x_2], \ldots, g_{n-1}[x_{n-1}, x_n])$. It is immediate that $g$ satisfies the hypothesis in Theorem 4.5. In addition, we have $\omega_1(g, \frac{1}{n}) \leq \omega_1(f, \frac{1}{n})$ and $m_f \leq m_g$. Furthermore, since $g$ is monotone on any interval of the form $[x_i, x_{i+1}]$, $i \in \{0, 1, \ldots, n-1\}$, we get $|g(x) - g(x)| \leq \omega_1(f, \frac{1}{n})$ for all $x \in [0, 1]$ as in the proof of Theorem 4.1. Taking into account the proof of Theorem 4.1, we get

$$|B^{(M)}_n(f)(x) - f(x)| = |B^{(M)}_n(g)(x) - f(x)| \leq |B^{(M)}_n(g)(x) - g(x)| + |f(x) - g(x)|$$

$$\leq \left( \frac{\omega_1(f, \frac{1}{n})}{m_f} + 3 \right) \omega_1(f, \frac{1}{n}) + \omega_1(f, \frac{1}{n})$$

$$= \left( \frac{\omega_1(f, \frac{1}{n})}{m_f} + 4 \right) \omega_1(f, \frac{1}{n}),$$

which proves the theorem. \(\square\)

**Corollary 4.7.** If $f: [0, 1] \to [0, \infty)$ is a strictly positive function satisfying the Lipschitz condition, then there exists a constant $C$ independent of $n$ and $x$ but depending on $f$, such that

$$|B^{(M)}_n(f)(x) - f(x)| \leq C \frac{n}{\sqrt{n}}, \quad x \in [0, 1], \quad n \in \mathbb{N}. $$
Proof. Since \( f \) satisfies the Lipschitz condition, it follows that there exists \( C_0 > 0 \) such that \( \omega_1(f, \frac{1}{n}) \leq \frac{C_0}{n} \). Substituting in (4.1) we obtain

\[
|B_n^{(M)}(f)(x) - f(x)| \leq \left( \frac{C_0}{m_f} + 4 \right) \frac{C_0}{n}, \quad x \in [0, 1].
\]

For \( C = \left( \frac{C_0}{m_f} + 4 \right) C_0 \) we get the desired conclusion. \( \square \)

Remarks. (1) Theorem 4.6 gives the order of uniform approximation (with the constant in \( O \) depending on \( f \))

\[
O\left\{ n \left[ \omega_1 \left( f, \frac{1}{n} \right) \right]^2 + \omega_1 \left( f, \frac{1}{n} \right) \right\},
\]

which, as it was pointed out in Introduction, for the classes of Lipschitz functions \( Lip \alpha \) gives the approximation order \( 1/n^{2\alpha-1} \), that for \( \alpha \in (2/3, 1] \) is essentially better than the general approximation order \( O[\omega_1(f, 1/\sqrt{n})] = O[1/n^{\alpha/2}] \).

(2) Comparing with the approximation error given by the linear Bernstein polynomials \( B_n(f)(x) \), the case when to get, for example, the order of approximation \( O(1/n^2) \) where we have to suppose that \( f' \) is a Lipschitz 1-function, we see that in the case of approximation by \( B_n^{(M)}(f) \), this order can be achieved under the less restrictive condition that \( f \) is a Lipschitz 1-function. This shows that the saturation class for the max-product Bernstein operator \( B_n^{(M)} \) differs (in fact it is much larger) from the saturation class for the linear Bernstein polynomials. The saturation problem for \( B_n^{(M)} \) will be studied in another paper.

Now, since \( B_n^{(M)}(f)(0) - f(0) = B_n^{(M)}(f)(1) - f(1) = 0 \), it is natural to look for a better pointwise estimate near to the endpoints 0 and 1. In this sense, we present the following two results.

Theorem 4.8. Let \( f: [0, 1] \to [0, \infty) \) be a continuous function. Then

\[
|B_n^{(M)}(f)(x) - f(x)| \leq 24 \omega_1 \left( f, \sqrt{\frac{x(1-x)}{n}} \right)
\]

for all \( x \in [0, 1/(n+1)] \cup [n/(n+1), 1] \) and \( n \in \mathbb{N}, n \geq 2 \).

Proof. First, let us choose arbitrary \( x \in [0, 1/(n+1)] \). By [1, Relation (4.3), p. 9], we have

\[
|B_n^{(M)}(f)(x) - f(x)| \leq \left( 1 + \frac{1}{2} B_n^{(M)}(\varphi_2)(x) \right) \omega_1(f; \delta), \quad (4.2)
\]

where

\[
\varphi_2(x) = \begin{cases} 
0 & \text{if } x < 1/(n+1), \\
2(n+1) & \text{if } x > n/(n+1), \\
2(n+1)x - 2nx & \text{otherwise}.
\end{cases}
\]
where \( \varphi_x(t) = |t - x|, \ t \in [0, 1] \) and \( \delta > 0 \) is chosen arbitrary. So, it is enough to estimate

\[
E_n(x) := B_n^{(M)}(\varphi_x)(x) = \frac{\displaystyle \sum_{k=0}^{n} p_{n,k}(x) \left| \frac{k}{n} - x \right|}{\displaystyle \sum_{k=0}^{n} p_{n,k}(x)}.
\]

Since \( x \in [0, 1/(n+1)] \), by Lemma 2.6 we get \( \sum_{k=0}^{n} p_{n,k}(x) = p_{n,0}(x) \), which immediately implies \( E_n(x) = \sum_{k=0}^{n} \left( \frac{k}{n} - x \right) \frac{p_{n,k}(x)}{p_{n,0}(x)} \). Let \( k_0 \in \{0, 1, \ldots, n\} \) be such that \( E_n(x) = \left( \frac{n}{k_0} \right) \left( \frac{x}{1-x} \right)^{k_0 - 1} \right) \frac{x}{n} \). If \( k_0 = 0 \) then \( E_n(x) = x \). If \( k_0 > 0 \) then we get

\[
E_n(x) = \left( \frac{n}{k_0} \right) \left( \frac{x}{1-x} \right)^{k_0} \frac{x}{n} = \left( \frac{n-1}{k_0-1} \right) \left( \frac{x}{1-x} \right)^{k_0} \frac{x}{n} \leq \left( 1 + \frac{x}{1-x} \right)^{-1} \frac{x}{n} \leq \left( 1 + \frac{x}{1-x} \right)^{-1} \frac{x}{1-1/(n+1)} = \frac{n+1}{n} \left( 1 + \frac{x}{1-x} \right)^{-1} \cdot x.
\]

Since the function \( g(x) = \frac{x}{1-x} \) is nondecreasing on \( (0, 1/(n+1)] \), we get

\[
E_n(x) \leq 2x \left( 1 + \frac{1/(n+1)}{1-1/(n+1)} \right)^{-1} = 2x \left( 1 + \frac{1}{n} \right)^{-1} \cdot \frac{n+1}{n} \leq 2ex.
\]

From the above estimates we get \( E_n(x) \leq 2ex \) for all \( x \in [0, 1/(n+1)] \). Now, taking \( \delta = 2ex \) in relation (4.2), we get

\[
|B_n^{(M)}(f)(x) - f(x)| \leq 2\omega_1(f, 2ex) \leq 12\omega_1(f, x),
\]

where we used the well-known property \( \omega_1(f, \lambda x) \leq ([\lambda] + 1)\omega_1(f, x) \). Because \( x \in [0, 1/(n+1)] \subset [0, 1/2] \) implies \( 1-x \geq 1/2 \), we get

\[
|B_n^{(M)}(f)(x) - f(x)| \leq 12\omega_1(f, x) \leq 24\omega_1 \left( f, \frac{1}{2} \cdot \sqrt{x} \cdot \sqrt{x} \right) \leq 24\omega_1(f, \sqrt{x(1-x)} \cdot \sqrt{x}) \leq 24\omega_1 \left( f, \sqrt{x(1-x)} \cdot \frac{1}{n} \right).
\]
Now, let us choose arbitrary \( x \in [n/(n+1),1] \). Take \( g: [0,1] \to \mathbb{R}, \ g(x) = f(1-x) \).

Because \( 1-x \in [0,1/(n+1)] \) and \( \omega_1(f, \sqrt{\frac{x(1-x)}{n}}) = \omega_1(g, \sqrt{\frac{x(1-x)}{n}}) \) and since \( |B_n^{(M)}(f)(x) - f(x)| = |B_n^{(M)}(g)(1-x) - g(1-x)| \) we immediately obtain the same estimate as in the previous case and the theorem is proved.

Combining Theorem 4.6 with Theorem 4.8, we obtain the following mixed pointwise-uniform estimate, essentially better near to 0 and 1.

**Corollary 4.9.** Let \( f: [0,1] \to [0,\infty) \) be a continuous and strictly positive function. Then, for all \( n \in \mathbb{N}, \ n \geq 2 \), we have the estimates:

\[
|B_n^{(M)}(f)(x) - f(x)| \leq 24 \omega_1 \left( f, \sqrt{\frac{x(1-x)}{n}} \right),
\]

for all \( x \in [0,1/(n+1)] \cup [n/(n+1),1] \), and

\[
|B_n^{(M)}(f)(x) - f(x)| \leq \left( \frac{24 \omega_1 \left( f, \frac{1}{n} \right)}{m_f} + 4 \right) \omega_1 \left( f, \frac{1}{n} \right),
\]

for all \( x \in [1/(n+1), n/(n+1)] \).

**Remark.** Since for \( x \in [0,1/(n+1)] \cup [n/(n+1),1] \) we easily have \( \sqrt{\frac{x(1-x)}{n}} \leq \frac{1}{n} \), even the uniform estimate generated in this way by Corollary 4.9, is obviously better than the uniform estimate in Theorem 4.6.

5. Quasi-Concavity Preserving Property

In the paper [1], it was proved that the Bernstein max-prod operator preserves the quasi-convexity. In this section we will prove that the discussed operator preserves the quasi-concavity too. We present the following shape preserving results.

**Theorem 5.1.** Let us consider the function \( f: [0,1] \to \mathbb{R}_+ \) and let us fix \( n \in \mathbb{N}, \ n \geq 1 \). Suppose, in addition, that there exists \( c \in [0,1] \) such that \( f \) is nondecreasing on \( [0,c] \) and nonincreasing on \( [c,1] \). Then, there exists \( c' \in [0,1] \) such that \( B_n^{(M)}(f) \) is nondecreasing on \( [0,c'] \) and nonincreasing on \( [c',1] \). In addition, we have \( |c-c'| \leq \frac{1}{n+1} \) and \( |B_n^{(M)}(f)(c) - f(c)| \leq \omega_1(f, \frac{1}{n+1}) \).

**Proof.** Let \( j \in \{0,1,\ldots,n\} \) be such that \( c \in \left[ \frac{j-1}{n+1}, \frac{j}{n+1} \right] \). We will study the monotonicity on each interval of the form \( \left[ \frac{j}{n+1}, \frac{j+1}{n+1} \right], \ j \in \{0,1,\ldots,n\} \). Then by the continuity of \( B_n^{(M)}(f) \), we will be able to determine the monotonicity of \( B_n^{(M)}(f) \) on \([0,1] \).

Let us choose arbitrary \( j \in \{0,1,\ldots,n\} \) and \( x \in \left[ \frac{j}{n+1}, \frac{j+1}{n+1} \right] \). By the monotonicity of \( f \), it follows that \( f \left( \frac{j}{n+1} \right) \geq f \left( \frac{j+1}{n+1} \right) \geq \cdots \geq f(0) \). By Lemma 2.1(ii), it easily
follows that $f_{j,n,j}(x) \geq f_{j-1,n,j}(x) \geq \cdots \geq f_{0,n,j}(x)$. Now, by Lemma 2.2 it follows that $B_n^{(M)}(f)(x) = \bigvee_{k=j}^n f_{k,n,j}(x)$. Since $B_n^{(M)}(f)$ is defined as the maximum of nondecreasing functions, it follows that it is nondecreasing on $[\frac{j}{n+1}, \frac{j+1}{n+1}]$. Taking into account the continuity of $B_n^{(M)}(f)$, it is immediate that $f$ is nondecreasing on $[0, \frac{1}{n+1}]$.

Now, let us choose arbitrary $j \in \{j_0 + 1, j_0 + 2, \ldots, n\}$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. By the monotonicity of $f$, it follows that $f(\frac{j}{n}) \geq f(\frac{j+1}{n}) \geq \cdots \geq f(1)$. By Lemma 2.1(i), it easily follows that $f_{j,n,j}(x) \geq f_{j+1,n,j}(x) \geq \cdots \geq f_{n,n,j}(x)$. Now, by Lemma 2.2 it follows that $B_n^{(M)}(f)(x) = \bigvee_{k=0}^n f_{k,n,j}(x)$. Since $B_n^{(M)}(f)$ is defined as the maximum of nonincreasing functions, it follows that it is nonincreasing on $[\frac{j}{n+1}, \frac{j+1}{n+1}]$. Taking into account the continuity of $B_n^{(M)}(f)$, it is immediate that $f$ is nonincreasing on $[\frac{j}{n+1}, 1]$.

Finally, let us discuss the case when $j = j_0$. If $\frac{j}{n} \leq c$, then by the monotonicity of $f$ it follows that $f(\frac{j}{n}) \geq f(\frac{\omega(j)}{n}) \geq \cdots \geq f(0)$. Therefore, in this case we obtain that $f$ is nondecreasing on $[\frac{j}{n+1}, \frac{j+1}{n+1}]$. It follows that $f$ is nonincreasing on $[0, \frac{j+1}{n+1}]$ and nonincreasing on $[\frac{j+1}{n+1}, 1]$. In addition, $c' = \frac{j+1}{n+1}$ is the maximum point of $B_n^{(M)}(f)$ and it is easy to check that $|c - c'| \leq \frac{1}{n+1}$.

We prove now the last part of the theorem. First, let us notice that $B_n^{(M)}(x) \leq f(c)$ for all $x \in [0, 1]$. Indeed, this is immediate by the definition of $B_n^{(M)}(f)$ and by the fact that $c$ is the global maximum point of $f$. This implies

$$|B_n^{(M)}(f)(c) - f(c)| = f(c) - B_n^{(M)}(f)(c) = f(c) - \bigvee_{k=0}^n f_{k,n,j_0}(c) \leq f(c) - f_{j_0,n,j_0}(c) = f(c) - f\left(\frac{j_0}{n}\right).$$

Since $c, \frac{j_0}{n} \in [\frac{j_0}{n+1}, \frac{j_0+1}{n+1}]$, we easily get $f(c) - f(\frac{j_0}{n}) \leq \omega_1(f, \frac{1}{n+1})$ and the theorem is proved completely.

**Corollary 5.2.** If $f : [0, 1] \to \mathbb{R}_+$ is continuous and quasi-concave on $[0,1]$, then for all $n \in \mathbb{N}$, $n \geq 1$, $B_n^{(M)}(f)$ is quasi-concave on $[0,1]$.

**Proof.** It is immediate by the remark after Definition 2.7 and by Theorem 5.1.
References

Article ID 590589, 26 pp.


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