Mathematics of inequality: in social sciences, economy and more

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1 What means "better"? What means "the best"? Comparing the uncomparable

1.1 Partial orders: leader property

Let Ω be a population composed of individuals having the same characteristics. We can compare them according to several numeric criteria: wealth, height, income,schooling, health, age, notoriety,etc. Thus any individual "i" may be characterized by a vector Z_i with d components, where d is the number o measured characteristics.

Thus the individuals can be compared among themselves on each component. The tentation of making tops is licit at unidimensional level. But in multidimensional case, the order relation is not total, usually the objects are not comparable. In spite of that, people want to compare them in order to make a decision. Sometimes it is unavoidable

Suppose that we know the probability distribution of the measured characteristics of the members of the population, each individual characterized by d characteritics .

In probabilistic terms, it is an idealization, meaning that we know the probability $F_Z(B) := P(Z \in B)$ if Z is a member of Ω , considered a d-dimensional vector and B is a d-dimensional Borel set.

Extract n individuals from the population;namely $(X_j)_{1 \le j \le n}$. We are interested in questions as:

- Which is the probability that this random set have a maximum? A minimum? Both a maximum and a minimum? If we pick a member of the population what is the probability that it is comparable with others? In general what is the probability that the population contain at least two comparable members?

Translate this in mathematical language

Let Z_k be a sequence of **iid** d-dimensional random vectors and let F be their common distribution.If we shall write only Z we shall understand that Z is a copy of Z_j.

(Of course we could replace "iid" by something else, but the situation is very difficult even so)

Warning! The same letter will be used both for distribution probability and for distribution function: if $B \subset \mathbb{R}^d$ is a borelian set then $F(B) = P(Z \in B)$; if $\mathbf{x} = (x_j)_{1 \le j \le d}$ is a vector from $\mathbb{R}^d$, then $F(\mathbf{x}) := P(Z \le \mathbf{x})$ or, explicitly,

$F(\mathbf{x}) := P(Z_1 \le x_1, Z_2 \le x_2, ..., Z_d \le x_d)$

$F^*(\mathbf{x}) := P(Z \ge \mathbf{x})$

Let also $\Phi(\mathbf{x}, \mathbf{y}) = P(\mathbf{x} \le Z \le \mathbf{y})$.

Let $S = S(F) = \text{supp}(F)$.

Precisely, $z \in S$ if and only if $\text{P}(|Z - z| < \varepsilon) > 0$ for any $\varepsilon > 0$.Of course S is closed: if a sequence $(z_n)_n$ is in S and $z_n \to z$ then z is in S, too. For $\text{P}(|Z - z| < \varepsilon) \ge P\left(|Z - z_n| < \frac{\varepsilon}{2}\right)$ if n is great enough.

Warning! In the 2-dimensional case we will prefer the notation $Z_k = (X_k, Y_k)$.

Now $F(x, y)$ means $P(X_k \le x, Y_k \le y)$. In this particular case the marginals will be denoted by F_X and F_Y : $F_X(x) = P(X_j \le x), F_Y(x) = P(Y_j \le x)$.

The main objects of interest:

(0.1) $a_n = P(\text{there exists } j \in \{1, .., n\} \text{ such that } Z_i \le Z_j \text{ for all } i)$

(0.2) $b_n = P(\text{there exists } j \in \{1, .., n\} \text{ such that } Z_i \ge Z_j \text{ for all } i)$

(0.3) $c_n = P(\text{there exists } i, j \in \{1, .., n\} \text{ such that } Z_i \le Z_k \le Z_j \text{ for all } k \ne i, j)$

Thus a_n is the probability that one of these n radom vectors be the greatest of them, b_n is the probability that one of them be the smallest and c_n the probability that the set $\{Z_1, ..., Z_n\}$ has both a minimum and a maximum..

Notice that all the probabilities a_n, b_n, c_n depend only on de distribution F of Z_j.

Example. $Z = (X, X+U), X\tilde{\ }U(0, 10), U\tilde{\ }U(0, 1)$

Definitions

Let a, b, c the inferior limits of these sequences.

If $a > 0$ we say that F has the leader property, or, by abuse, that the sequence Z has the leader property.

If $b > 0$, F (or Z) has the min property

If $c > 0, F$ (or Z) has the order property

Obvious facts

(0.4) $c_n \le \min(a_n, b_n)$

(0.5) If Z has the leader property, then $f(Z)$ has again the leader property for any increasing $f : \mathbb{R}^d \to \mathbb{R}^d$ and the same holds for the other two properties

Remark. Here "increasing" means **a nondecreasing mapping** having the property that $f(\mathbf{z}) \le f(\mathbf{z}') \iff \mathbf{z} \le \mathbf{z}'$.Typical examples of increasing mappings are $f(\mathbf{z}) = (f_j(z_j))_{1\le j\le d}$ with $f_j : I_j \to \mathbb{R}$ increasing. Example: For $d = 2$ the mappings $f(x,y) = (-\ln(1-x), -\ln(1-y))$ or $f(x,y) = \left(\frac{1}{1-x}, \frac{1}{1-y}\right)$ are increasing.

(0.6) The probabilities a_n, b_n, c_n remain the same if we replace the sequence $(Z_n)_n$ with $\zeta_n = (f(Z_n))_n$ with f increasing.

(0.7) If Z has a leader and $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is increasing then $\phi(Z)$ has a leader, too. Moreover, if Z has a leader and ϕ is decreasing, then $\phi(Z)$ has the min property.

Definition 2. A set S **has a leader** if it contains an increasing sequence of points $(z_n)_n$ such that $S \le \lim_n z_n$. Clearly that if S is compact then S has a leader if there exists a point $z_0 \in S$ such that $S \le z_0$. This leader z_0 is unique. The set has a **weak leader** z^*if there exists no $z \in S$ such that $z^* < z$. In the same way S has a minimum if if it contains a decreasing sequence of points $(z_n)_n$ such that $\lim_n z_n \le S$.

Fact : If S is compact then S has no leader if and only if it has at least two weak leaders. Indeed, as S is compact then $S \subset S_1 \times S_2$ where $S_j = proj_j(S)$ are compact sets. Let $b_j = \sup S_j$. The sections $C_1 = \{y : (b_1, y) \in S\}, C_2 = \{x : (x, b_2) \in S\}$ are compact and $z^* = (b_1, \sup C_1)$ and $z^{**} = (\sup C_2, b_2)$ are two weak leaders. They are different, since if $z^* = z^{**}$ then S would have a leader, which we denied.

Examples.

0. $S = \{(2,0), (1,1), (0,2)\}$ has no leader. All the points of S are weak leaders.

1.If $F = U(0,1) \otimes Q$ where $Q(x) = U(f(x), g(x))$ where $f, g : [0,1] \to \mathbb{R}$ are measurable and $f \le g$.Then $S = \mathrm{Supp}(F) = Cl\left\{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, f(x) \le y \le g(x)\right\}$. Here $Cl(A)$ means the closure of A. If g is nondecreasing then S has the leader $(1, g(1))$ and the minimum $(0, f(0))$

2.If F =Uniform(C) where $C \subset \mathbb{R}^2$ is a compact set such that $0 < \lambda^2(C) < \infty$.The leader is the point $(\max C_1, \max C_2)$ provided that it belongs to C.

1.1.1 1. The discrete case

Proposition 1 *1.1.*

a. If Z is discrete and $S = Supp(F)$ has a leader then the distribution F has the leader property, too. In this case $\lim_n a_n = \lim_n b_n = \lim_n c_n = 1$.

Otherwise written, the occurence of a leader is unavoidable.

b. If S has at least two incomparable weak leaders, then Z has no leader, too.

c. If, moreover, S is compact, then F has the leader(minimum) *property if and only if S has a leader* (minimum).

1.1.2 2. The continuous case

It is much more involved

Suppose that F has the property that $F(H) = 0$ for any hyperplane from $\mathbb{R}^d$ (for instance if F is absolutely continuous with respect to Lebesgue measure λ^d).

FACT

(2.1) $a_n = nP(Z_j \le Z_n \forall j \in \{1, 2, ..., n\})$

(2.2) $b_n = nP(Z_j \ge Z_n \forall j \in \{1, 2, ..., n\})$

(2.3) $c_n = n(n-1)P(Z_i \le Z_n \le Z_j, \forall i, j \in \{1, 2, ..., n\})$

Computation rules.

1.1.3 Lemma 2.2

Let $(\Omega, \mathcal{K}, P)$ be a probability space $(E, \mathcal{E})$and $(H, \mathcal{H})$be measurable spaces. Let $Z = (X, Y) : \Omega \to E \times H$ be a random variable. Suppose that the distribution of Z is decomposable, meaning that it can be written as

(A) $F_Z = F_X \otimes F_{Y|X}$ where F_X is a probability on $(E, \mathcal{E})$ and the conditioned distribution $F_{Y|X}$ is a transition probability from $(E, \mathcal{E})$ to $(H, \mathcal{H})$

Let $f : E \times H \to \mathbb{R}$ be measurable and bounded. Then

(B) $\mathrm{E}(\phi(Z)|X)) = \int \phi(X, y)\, dF_{Y|X}(y)$

In the particular case when X and Y are independent, $F_Z = F_X \otimes F_Y$ hence

(C) $\mathrm{E}(\phi(Z)|X)) = \int \phi(X, y)\, dF_Y(y)$

Proposition 2.2. If $(Z_k)_k$ are iid $F-$ distributed, then

(A) $P(Z_j \le Z_n \forall j \in \{1, 2, ..., n\})$=$\mathrm{E}F(Z)^{n-1}$ holds.

(B) Suppose moreover that F has the property that $F(H) = 0$ for any hyperplane from $\mathbb{R}^d$ (for instance if F is absolutely continuous with respect to Lebesgue measure λ^d). Then

(2.4). $a_n = n\int F^{n-1}dF$

(2.5) $b_n = n\int (F^*)^{n-1}\, dF$

(2.6) $c_n = n(n-1)\int\int \Phi^{n-2}(x, y)\, 1_{\{x \le y\}} dF(y)\, dF(x)$

In the 2-dimensional case: if $F = F_Z$ is absolutely continuous and has the density p then (2.4') $a_n = n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^{n-1}(x, y)\, p(x, y)\, dydx$, $b_n = n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (F^*)^{n-1}(x, y)\, p(x, y)\, dydx$..

To compute c_n is a difficult task because there are too many integrals

A negative result:

Proposition 2.4. Suppose that $(Z_n)_n$ is a sequence of iid bounded continuous random vectors F-distributed and the support of F is the compact S.

If S has no leader then F has no leader, too.

1.1.4 2.1. The absolutely continuous case

We start with a sufficient condition. To simplify the things,suppose that $d = 2$, $Z_n = (X_n, Y_n)$ and $F = F_X \otimes Q$ where Q is a transition probability from $\mathbb{R}$ to $\mathbb{R}$ having the meaning that $Q(x, B) = P(Y \in B|X = x)$ or, explicitly, that $\mathrm{E}u(Y|X) = \int u(X, y)\, Q(Y, dy)$ for any u bounded and measurable. Suppose that F_X is absolutely continuous with respect to Lebesgue measure and let ρ be its density: $dF_X(x) = \rho(x)\, dx$.

Let n be fixed. Sort the random variables $(X_j)_{1\le j\le n}$ as $X_{()} = (\ X_{(1)} \le X_{(2)} \le ... \le X_{(n)})$. Let $X^* = X_{(n)}$, $X^{**} = X_{(n-1)}, X_* = X_{(1)}, X_{**} = X_{(2)}$

The following facts are well known and can be found in any handbook of order statistics:

α. The density of $X_{()}$ is $p(x_1, ..., x_n) = n!\rho(x_1)\rho(x_2)...\rho(x_n)\, 1_{\{x_1\le x_2\le...\le x_n\}}$

β. The density of (X^{**}, X^*) is $p(x, y) = n(n-1)\, F_X^{n-2}(x)\, \rho(x)\, \rho(y)\, 1_{(x,\infty)}(y)$

γ. The density of (X_*, X_{**}) is $p(x, y) = n(n-1)\, \overline{F}_X^{n-2}(y)\, \rho(x)\, \rho(y)\, 1_{(x,\infty)}(y)$

δ. The density of X_{**} is $p(y) = n(n-1)\, \overline{F}_X^{n-2}(y)\, F_X(y)\, \rho(y)$

ε. The density of X^{**} is $p(x) = n(n-1)\, F_X^{n-2}(x)\, \overline{F}_X(x)\, \rho(x)$

Lemma 2.5

A. $\mathrm{P}(X^* - X^{**} > t) = \mathrm{E}\frac{\overline{F}_X(X^{**}+t)}{\overline{F}_X(X^{**})} = n(n-1)\int_{-\infty}^{\infty} \overline{F}_X(x+t)\, F_X^{n-2}(x)\, \rho(x)\, dx$

B. $\mathrm{P}(X_{**} - X_* > t) = \mathrm{E}\frac{F_X(X_{**}-t)}{F_X(X_{**})} = n(n-1)\int_{-\infty}^{\infty} F_X(y-t)\, \overline{F}_X^{n-2}(y)\, \rho(y)\, dy$

As a consequence , if X is not bounded above (meaning that $\overline{F}_X(t) > 0$ for any real t),then $\mathrm{P}(X^* - X^{**} > t) > 0$ for any t and if it is not bounded below (meaning that $F_X(t) > 0$ for any t), then $\mathrm{P}(X_{**} - X_* > t) > 0$ for any t. Thus, if X is unbounded both below and above, $P\left(X_{(2)} - X_{(1)} > t\right)$ and $P\left(X_{(n)} - X_{(n-1)} > t\right)$ are positive for all t.

A result for non-negative random variables.

If they are thought as waiting times, a useful tool is the concept of *hazard rate*, also known as *failure rate*.

Definition. Let $X > 0$ be absolutely continuous, $\overline{F_X} = 1 - F(x)$ be its tail and ρ its density. The quantity $\lambda_X = \frac{\rho}{\overline{F_X}}$ is called the hazard rate of X (or of F_X). Then $\overline{F_X}(x) = e^{-\int_0^x \lambda_X(u)du}$. If λ_X is non-decreasing one says that X is of type IFR (Increasing Failure rate) and if it is non-increasing X is of type DFR (Decreasing Failure Rate). Usually one and writes $X \in$ IFR in the first case and $X \in$ DFR in the second one. (A better notation would be, of course, $F_X \in IFR/DFR$ but this is the tradition) . Of course $X \in IFR \cap DFR$ means that X is exponentially distributed. It is easy to see that the mapping $x \longmapsto {}_tp_x$ is non-increasing in the IFR case and non-decreasing in the DFR case. Moreover, all the random variables of type DFR are not bounded above.

Proposition 2.6 Let $Z_n = (X_n, Y_n)$ be non-negative iid random vectors having the distribution $F = F_X \otimes Q$. Suppose that

(i). $Q(x, [x-m, x+M]) = 1$ for some nonnegative m, M for all $x \geq 0$ and

(ii). $\sup_{t \in Supp(F_X)} \lambda_X(t) = \lambda_0 < \infty$

Then $a = \lim a_n \geq e^{-(M+m)\lambda_0} > 0$ hence F has the leader property.

As a particular case, if $X \in DFR$ then $\lambda_0 = \lambda_X(0) > 0$ thus $a \geq e^{-(M+m)\lambda(0)}$

Remark. Usually one writes "$(Y|X) \subset [X-m, X+M]$ a.s." instead of (i).

The result can be extended to random variables which are unbounded both above and below.

Corollary 2.7 If both X_+ and X_- are DFR then F has both the leader property and the min property.

OPEN QUESTION. We were not able to answer the question: If F has both the leader property and the min property is it true or not that F has the order property? We believe that the answer should be affirmative.

Example 2.8. Exponential versus uniform Z$= (X, Y), X\tilde{~}Exp(1), X+U, U\tilde{~}Uniform(0,1), X$ independent on U. Z_n are independent copies of Z. Then X is DFR, $X \leq Y \leq X+1$ and, according to Proposition 2.6 with $m = 0, M = 1$ we see that $a \geq e^{-1} = 0.367\,88$.

As the density of Z is $p(x,y) = e^{-x} 1_{(x,x+1)}(y)$, the exact limit is

$a = \lim_n \int_0^\infty e^{-x} \int_x^1 nF^{n-1}(x,y)\, dydx$ with

$F(x,y) = 1 - e^{-(y-1)_+} + \left(x - (y-1)_+\right) e^{-x}$

At this stage we do not know if a is computable or not. Computer simulations suggest that a is much greater: it seems that $a \geq 0.8$

In this very case we can transform the random vector Z into a bounded one having the same ordering property. Let $\phi = F_X^{-1}$. In our case $\phi(x) = -\ln(1-x)$. The vector $\zeta = (F_X(X), F_X(Y)) := (\xi, \eta)$ has the support in $[0,1]^2$ and its density is

$\pi(s,t) = \frac{1}{1-t} 1_{\left\{s<t<1-\frac{1-s}{e}\right\}}$. Notice that the density of ξ is standard uniform and that π is unbounded.

It is easy to prove that the density of $Z = (F_X(X), F_X(Y))$ is unbounded if $X \in$ DFR.

One may ask if in the bounded case - when $Supp(F)$ is compact - is it necessary that the density of Z be unbounded. The answer is NO.

Example 2.9. Uniform in $A_\alpha = \{(x, y) : 0 \leq x \leq 1, f(x) \leq y \leq g(x)\}, f, g$ increasing 1. $f(x) = x, g(x) = mx+n, m+n = 1 :$ F has the leader property if and only if $m < 1$.

2. $f(x) = x, g(x) = \frac{1+x^2}{2}$: F has STRONG leader property: $a = 1$!

3. CONJECTURE. If $g'(1) = f'(1), f(1) = g(1)$ then $a = 1$

Fig 2. Black: case $m = .75$. Red: $g(x) = \frac{1+x^2}{2}$

Example 2.10. Mixture of small uniforms. Compare the probabilitic method and the analytic one Let $0 = \alpha_0 < \alpha_1 < ... < \alpha_n < ...$ and $0 = \beta_0 < \beta_1 < ... < \beta_n < ...$ be such $\lim \alpha_n = \lim b_n = 1$.Let $(p_j)_{j\geq 1}$ be a distribution probability on the set of positive integers such that $p_j > 0$ for all j.Let $I_k = (\alpha_{k-1}, \alpha_k), J_k = (\beta_{k-1}, \beta_k)$, $k \geq 1$ and finally, let $Z = (X, Y)$ be a random vector with the distribution $F =_{k=1}^{\infty} p_k \text{Uniform}(I_k \times J_k) =_{k=1}^{\infty} p_k \text{Uniform}(I_k) \otimes \text{Uniform}(J_k)$ Fig 3. The set A is a union of squares

Its density is
$p =_{k=1}^{\infty} \frac{p_k}{(\alpha_k - \alpha_{k-1})(\beta_k - \beta_{k-1})} 1_{I_k \times J_k}$ and the marginal densities are
$p_X =_{k=1}^{\infty} \frac{p_k}{(\alpha_k - \alpha_{k-1})} 1_{I_k}$
$p_Y =_{k=1}^{\infty} \frac{p_k}{(\beta_k - \beta_{k-1})} 1_{J_k}$

Bound given by Proposition 2.6 : $a \geq \exp\left(-\sup_{k\geq 1} \frac{p_k}{p_{k+1} + .p_{k+2} + ...}\right)$

If $\sup_{k\geq 1} \frac{p_k}{p_{k+1} + .p_{k+2} + ...} < \infty$ (or, which is the same, if $\inf_{k\geq 1} \frac{p_{k+1} + .p_{k+2} + ...}{p_k} > 0$) then Z has the leader property.

Analytic approach : estimate $a = \lim \mathrm{E} n F^{n-1}(\zeta)$ using brute force:
$\mathrm{E} n F^{n-1}(\zeta) =_{k=1}^{\infty} \int_0^1 \int_0^1 n (p_1 + ... + p_k + p_{k+1} st)^{n-1} p_k dtds$
$\geq_{k=1}^{\infty} n (p_1 + ... + p_k)^{n-1} p_k \geq \inf_{k\geq 1} \frac{p_{k+1}}{p_k}$

Conclusion

$$a \geq \max\left(\exp\left(-\sup_{k\geq 1} \frac{p_k}{p_{k+1} + .p_{k+2} + ...}\right), \inf_{k\geq 1} \frac{p_{k+1}}{p_k}\right)$$

For the last inequality we have used the following elementary result:

Lemma 2.11 Let $0 = \alpha_0 < \alpha_1 < \alpha_2 < ....$ be an increasing sequence such that $\lim \alpha_k = 1$. Let $p_k = \alpha_k - \alpha_{k-1}, k \geq 1$ and $\varepsilon = \inf_{k\geq 2} \frac{p_k}{p_{k-1}}$

Then $\sum_{k=1}^{\infty} (\alpha_k - \alpha_{k-1}) n \alpha_{k-1}^{n-1} \geq \varepsilon$

Now we have a clue to decide if Z has a leader: if $Y - X$ is bounded and X is unbounded. But what can we say if $Y - X$ is unbounded, too?

One is obliged to use the brute force.

A result that may help

Lemma 2.12.

1. Let $f:[0,1]\to[0,\infty)$ be continuous at $x=1$. Then $\lim\limits_{n} n\int_0^1 x^{n-1}f(x)\,dx = f(1)$

2. Let $G:[0,1]\to[0,1]$ be increasing and differentiable such that $G(1)=1$ and let f as above.

Then $\lim\limits_{n} n\int_0^1 G^{n-1}(x)\,f(x)\,dx = \frac{f(1)}{G'(1)}$

Here are two cases when this Lemma is applied:

Example 2.13. Exponential versus Exponential Suppose that $X\tilde{}Exp(1), U\tilde{}Exp(\lambda), Y = X +$

$$Z=(X,Y)=(X,X+U)$$

Then

Proposition 2 *2.14. Suppose that $\lambda>1$ is a positive integer.Let F_λ be the distribution of Z,*

1. $\liminf\limits_{n\to\infty} a_n(\lambda)\geq\frac{\lambda-1}{\lambda}$ hence F_λ has the leader property
2. If $\lambda=2$,$\lim a_n=\ln 2$
3. The sequence $(a_n(\lambda))_{\lambda\geq 2,\lambda\in\mathbb{N}}$ is increasing. Thus $\lambda\geq 2 \implies a(\lambda)\geq\ln 2$
4. If $\lambda=1$ then $\lim a_n(\lambda)=0$ hence F_1 has NOT the leader property

QUESTION. WHAT IF λ is not an integer?

If we use the transform $x\longmapsto 1-e^{-x}$ we get the vector $\zeta=\left(1-e^{-X},1-e^{-Y}\right)$

Its density is $\rho(s,t)=\frac{\lambda(1-t)^{\lambda-1}}{(1-s)^{\lambda}}$ and its distribution function for $0<s<t<1$ is $G(s,t)=s-(1-t)^{\lambda}\frac{(1-s)^{\lambda-1}-1}{\lambda-1}$. We get

Corollary The distributions F_λ on $[0,1]^2$ with the densities $p^*_\lambda(x,y)=$

$$\begin{cases}\lambda(1-x)^{-\lambda}(1-y)^{\lambda-1}1_{\{0<x<y<1\}} & if\quad \lambda>1\\ \frac{1}{1-x}1_{\{0\leq x\leq y\leq 1\}} & if\quad \lambda=1\end{cases}\quad ,\lambda \text{ positive integer}$$

have the leader property for $\lambda\geq 2$ and not for $\lambda=1$

Example 2.15. A puzzling leaderles distribution Let $X\tilde{}U(0,1), U\tilde{}U(0,\alpha), \alpha>0, Y=X+U$,

$$Z=(X,Y)=(X,X+U)\tilde{}\ \text{Uniform}\ (A)$$

Figure 4 The set A is between the two segments. Here $\alpha=0.2$

$$a_n=\frac{1}{a}\int_0^1\int_x^{x+\alpha} n\left(x-\frac{(x+\alpha-y)_+^2-(\alpha-y)_+^2}{2\alpha}\right)^{n-1}dydx \underset{n\to\infty}{\longrightarrow} 0$$

The proof is not simple. To conclude:

Proposition 2.17 The distribution $F = U(0,1) \otimes Q$ with $Q(x) = U(x, x+\alpha)$ has **never** the leader property.

Example 2.18. An exact result.Mixture of copulae $F_\alpha = \alpha\text{Uniform}(D) +$

$\beta\text{Uniform}(E)$ where D is the first diagonal of the unity square and E is the second one. Here $\alpha + \beta = 1, \alpha, \beta \geq 0$

Or, F_α is a mixture between the monotonic and the antimonotonic copulae. Its distribution function is

(2.14) $F(x,y) = \alpha \min(x,y) + \beta(x+y-1)_+$

The computations yield

$a_n = n\text{E}F^{n-1}(Z) = \frac{\alpha^n + 2\alpha^{n-1}\beta}{2^n} + \frac{\alpha}{1+\beta}\left(1 - \left(\frac{1-\beta}{2}\right)^n\right)$ hence

(2.15) $a = \lim a_n = \frac{\alpha}{1+\beta}$

For $\alpha = \beta = \frac{1}{2}$, $a_n = \frac{3}{2^{2n}} + \frac{1}{3}\left(1 - \frac{1}{2^{2n}}\right) \to a = \frac{1}{3}$

In the same way

(2.16) $b = a = \frac{\alpha}{1+\beta}$

(2.17) $c = \alpha^2/2$

1.2 3. Mixtures of comonotonic distributions

Definition. We say that the random vector $Z = (X,Y) \subset [0,\infty)^2$ is **comonotonic** if $Y = f(X)$ for some nondecreasing f. Then the distribution F can be written as $F = F_X \otimes Q$ where $Q(x) = \delta_{f(x)}$. Call this type of distribution "$F_X - f$". Its support is $Graph(f)$. Of course the leader is $(1, f(1))$.

Example 3.1. $F = U \otimes \delta_f$ whith $U = Uniform(0,1), f : [0,1] \to [m, M]$

increasing and continuous is the distribution of the vector $Z = (X, f(X))$ where X is uniformly distributed on $[0,1]$.

Let ϕ be its pseudo inverse defined as $\phi(x) = f^{-1}(x)$ if $x \in [m, M], \phi(x) = 0$ if $x < m$, $\phi(x) = 1$ if $x > M$.

Then the distribution function of Z is

(3.1) $F(x_1, x_2) = \min\left(x_1, f^{-1}(x_2)\right)$ for $x_j \in [0,1]$.

and the computation rules are $\text{E}u(Z) = \text{E}u(X, f(X)) = \int_0^1 u(x, f(x))\,dx$ thus

a. $\text{E}u(Z) = \int u dF = \int_0^1 u(x, f(x))\,dx$ if $u : [0,1] \times \mathbb{R} \to \mathbb{R}$ is bounded and measurable

b. $\text{E}v(Z_1, Z_2) = \int v dF^2 = \int_0^1 \int_0^1 v(x, f(x), y, f(y))\,dydx$ if $v : ([0,1] \times \mathbb{R})^2 \to \mathbb{R}$ is bounded and measurable and Z_j are iid $F - f$ distributed

In this is the trivial case it is obvious that

(3.2) $a_n = b_n = c_n = 1$

Recall that we have denoted $\Phi(x,y) = P(x \leq Z \leq y)$

In this trivial case

$\Phi(x,y) = P(x_1 \leq X \leq y_1, \phi(x_2) \leq X \leq \phi(y_2)) = (\max(x_1, \phi(x_2)) \leq X \leq \min(y_1, \phi(y_2)))$

Thus

(3.3) $\Phi(x,y) = (\min(y_1, \phi(y_2)) - \max(x_1, \phi(x_2)))_+$

As $\Phi(x, f(x), y, f(y)) = (\min(y, \phi(f((y))) - \max(x, \phi(f(x))))_+ = (y-x)_+$ we get according to b.

(3.4) $\quad c_n = n(n-1)\int_0^1 \int_0^1 (y-x)_+^{n-2}\, dydx = n(n-1)\int_0^1 \int_0^{1-x} t^{n-2} dtdx = 1$

Next case is

Example 3.2. Mixture of comonotonic distributions. Let $p, q \in [0,1], p+q = 1$

$F = pF_1 + qF_2, F_1 = U \otimes \delta_f, F_2 = U \otimes \delta_g$ with f, g continuous increasing and U the standard uniform. Figure 5. $f(x) = (3x+1)/4, g(x) = (1+x^2)/2$

The result is

Proposition *1. Suppose that f and g are continuous and differentiable.

$a = \begin{cases} q & if\ f(1) < g(1) \\ q + \frac{p}{p + q\frac{f'(1)}{g'(1)}} & if\ f(1) = g(1) \end{cases}$. Notice that $a = 1$ if $f(1) = g(1)$ and $f'(1) = g'(1)$

$b = \begin{cases} p & if\ f(1) < g(1) \\ p + \frac{q}{q + p\frac{g'(0)}{f'(0)}} & if\ f(1) = g(1) \end{cases}$ .Notice that $b = 1$ if $f(0) = g(0)$ and $f'(0) = g'(0)$

To find c is a challenge. All we know is

$\frac{p^2}{2} + pq < c < \min\left(p, q + \frac{p}{p+2q}\right)$

Generalization for N functions Proposition 2.4.

Let $N \geq 2, (F_i = \rho_i \cdot \lambda)_{1 \leq i \leq N}$ be probability distributions on $[0,1], (p_j)_{1 \leq j \leq N}$ be a probability distribution on $\{1, ..., N\}$. Suppose that $\rho_i, p_i > 0$. Let $f_i : [0,1] \to [0, \infty)$ be increasing and differentiable, $g_i = f_i^{-1}$.Let also $\alpha_i = f_i(1)$.Thus $g_i(\alpha_i) = 1$

Suppose that there exists $\varepsilon > 0$ and $1 \leq k \leq N$ such that $\left(f_i|_{[1-\varepsilon,1]}\right)_{1 \leq i \leq N}$ is a non-decreasing finite sequence of functions and, moreover, that $\left(g_i|_{[1-\varepsilon,1]}\right)_{1 \leq i \leq N}$ is non-increasing and

$\alpha_1 \leq ... \leq \alpha_k < \alpha_{k+1} = \alpha_{k+2} = ... = \alpha_N = 1$

Finally, let $F = \sum\limits_{j=1}^{N} p_j F_j \otimes \delta_{f_j}$

Then

(2.8) $\quad \lim_n a_n = \sum\limits_{i=k+1}^{N} \frac{p_i \rho_i(1)}{\sum\limits_{j=1}^{i} p_j \rho_j(1) + \sum\limits_{j=i+1}^{N} p_j \rho_j(g_j(\alpha_i)) g_j'(\alpha_i) f_i'(1)}$