ON ORDER CONTINUOUS BANACH C(K)-MODULES

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Abstract

Let X be a Banach space and let X' be the norm dual of X. $C(K)$ denotes the Banach space of all continuous real or complex valued functions on a compact Hausdorff space K with the supremum norm. Suppose that X is a Banach $C(K)$-module. In this paper, we characterize the order continuity of Banach $C(K)$-module if dual Banach space X' does not contain the space ℓ^∞ of all bounded sequences by using Arens multiplication..

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1 Introduction

Lozanovsky's a study in [8] is that a Dedekind complete Banach lattice has order continuous norm if and only if it does not contain a copy of ℓ^∞.A. Kitover et al [6] investigate in a paper that firstly what is an analogue of Dedekind completeness for Banach $C(K)$-modules and secondly what should be considered as an analogue of order continuity for Banach $C(K)$-modules,[10]. The positive answer to the question in the first case is provided by the well-known notion of Kaplansky module by Kitover et al [6].Positive answer to the second assertion is given by again Kitover et al [6].

In this paper , we present the dual versions of these solutions. Namely, we take dual Banach $C(K)$- modules instead of Banach $C(K)$-modules.

For unexplained notion and terminology in this paper we refer to the books [1] and [7], [9].

Definition 1. [1] Let X be a Banach space and let $C(K)$ be the Banach space of all real or complex valued continuous functions on a compact Hausdorff space K. A bilinear mapping $b : C(K) \times X \to X$ is called a Banach $C(K)$-module if the following conditions are satisfied:

i.$a.(b.x) = (a.b).x$ for all $a, b \in C(K)$ and $x \in X$,

ii. $1.x = x$ for all $x \in X$, where 1 is the constant one function,

iii. The mapping b is continuous, i.e.,the inequality

$$\|a.x\| \leq \|a\| \|x\|$$

holds for every $a \in C(K)$ and $x \in X$.

We can extend the bilinear mapping b to dual spaces X' and X'' by using Arens multiplication, [2].

$$b' : X' \times X \to C(K)'$$

is defined by $b'(x', x)(a) = x'(b(a, x))$.

$$b'' : C(K)'' \times X' \to X'$$

is defined by $b''(\hat{a}, x')(x) = \hat{a}(b'(x', x))$.

$$b''' : C(K)'' \times X'' \to X''$$

is defined by $b'''(\hat{a}, x'')(x') = x''(b''(\hat{a}, x'))$.

In these mappings , if we put $X = C(K)$, then $C(K)'' = C(S)$ where S is a hyperstonian compact Hausdorff space.Let B be the Boolean algebra of all idempotents in $C(S)$. By bilinear mapping b defines a continuous map $m : C(K) \to L(X)$ defined by $m(a)x = a.x = b(a, x)$. Also, bilinear mapping b'' defines a continuous mapping $m^* : C(K)'' \to L(X')$ defined by $m^*(a)x' = a.x'$.

The properties of mappings m and m^* are given by the following Theorem.

Theorem 2. *Let X be a Banach $C(K)$-module and let $m : C(K) \to L(X)$ be a continuous mapping from $C(K)$ to the space $L(X)$ of all bounded linear operators from X into X. Then , the following claims are true:*

i. m is an algebra homomorphism.

ii. The mapping m^* satisfies: $m^*(a) = (m(a))^*$ for all $a \in C(K)$, where $(m(a))^*$ is the adjoint of $m(a)$.

iii. $m(1) = I$, where I denotes the identity operator.

iv. m^* is continuous from $C(K)''$ into $L(X')$ defined by $m^*(a)(x') = a.x'$.

v. X' is a Banach $C(K)''$-module.

vi. X'' is a Banach $C(K)''$-module.

vii. m^* is a weak to weak* continuous mapping from $C(K)''$ to X'.*

viii. m^* is an algebra homomorphism.

Proof. Proofs are done directly by using Arens extension mappings and module.

□

Let X be a Banach $C(K)$-module. Then, we can define an equivalent norm on X',

$$\|x'\|_1 = sup\{\|m^*(a)x'\| : a \in C(S), \|a\| \leq 1\}.$$

The homomorphism m^* is a contraction with respect to new norm. The $kerm^*$ is a closed ideal in $C(S)$, because of algebra homomorphism m^*. We can replace $C(S)$ by $C(T) \cong C(S)/kerm^*$, where $T = \{t \in S : a(t) = 0 \forall a \in kerm^*\}$. So, we can assume that m^* is one to one.

Theorem 3 (3,4). *Let X be a Banach $C(K)$-module and let $m^* : C(K)'' \to L(X')$ be a contractive homomorphism. Then,*

i.If for every $a, b \in C(K)''$, $|a| \leq |b|, then \|m^*(a)x'\| \leq \|m^*(b)x'\|$ for any $x' \in X'$.

ii. If m^* is one to one, then it is an isometry.

Definition 4 (4,6). Let $m : C(K) \to L(X)$ be a bounded unital algebra homomorphism and let $x \in X$. An idempotent $e_x \in B$ is said to be a carrier projection of x if $m(e_x)x = x$ and $e_x \leq e$ in $C(K)$, whenever $e \in B, m(e)x = x$

Theorem 5 (6). *Let X be a Banach $C(K)$-module and let m^* : $C(K)'' \to L(X')$ be a continuous unital algebra homomorphism. Let $x' \in X'$ and $e_{x'}$ be the carrier projection of x'. Then, for any $a \in C(K)''$ and $m^*(a)x' = 0$ if and only if $ae_{x'} = 0$.*

Proof. Let $ae_{x'} = 0$.Then, $m^*(a)x' = m^*(a)m^*(e_{x'})x' = m^*(ae_{x'})x' = 0$.

Assume that for some $e \in B, m^*(e)x' = 0$. Then, $m^*((1-e)x') = x'$ and therefore $e_{x'} \leq 1 - e$. So, $ee_{x'} = 0$. Suppose that for some $0 \leq a \in C(K)''$. We have $m^*(a)x' = 0$ and that for some $t \in S, a(t) > 0$. Since S is totally disconnected, there are some $\epsilon > 0$ and $e \in B$ such that $e(t) = 1$ and $0 \leq \epsilon e \leq a$. From here we get $\epsilon\|m^*(e)x'\| \leq \|m^*(a)x'\| = 0$. Therefore, $ee_{x'} = 0$ and $e_{x'}(t) = 0$ and hence $ae = 0$.

If $m^*(a)x' = 0$ for some $a \in C(K)''$, then $m^*(|a|)x' = 0$ hence $|a|e_{x'} = 0$ and so $ae_{x'} = 0$

□

Definition 6 (6). A Banach space $C(K)$ -module X is called a Veksler module if any $x \in X \setminus \{0\}$ has a carrier projection $e_x \in C(K)$.

A compact Hausdorff space K is called quasi-Stonian if it is basically disconnected , that is, the closure of every open G_δ set in K is open. It is well-known that the following claims are equivalent:

a. K is quasi-Stonian,

b. $C(K)$ is σ-Dedekind complete,

c.Every non-negative sequence bounded from above in $C(K)$ has a supremum in $C(K)$.

d. Every principal band in $C(K)$ is a projection band.

Definition 7 (6). A Banach $C(K)$-module X is said to be a Kaplansky module if it satisfies the following conditions:

i. The compact space K is Stonian,

ii. For any $x \in X$ and for any non-negative set $\{a_\alpha\}$ bounded above in $C(K)$ the following holds: if $a_\alpha x = 0$ for all α, then $ax = 0$, where $a = sup_\alpha a_\alpha$.

Theorem 8 (Lozanovsky,5, 6,8). *Let E be a σ- Dedekind complete Banach lattice . The following are equivalent:*

i. The original lattice norm on E is order continuous.

ii. E does not contain ℓ^∞ as a closed subspace.

iii.E does not contain ℓ^∞ as a closed sublattice.

Definition 9 (4,5,6). Let B be a Boolean algebra of projections in $L(X)$. B is called a Bade-complete Boolean algebra of projections if for any increasing net (e_α) in B and for every $x \in X$ we have $lim_\alpha \|(e - e_\alpha)x\| = 0$, where $e = sup_\alpha e_\alpha$.

Let B be idempotents in $C(K)'' = C(S)$. B consists of characteristic functions of the clopen subsets of S. And $m^*(B) = B^*$ is a Bade -complete Boolean algebra of projections on X'.

2 The cyclic Banach spaces

In this section we define cyclic Banach space by using Banach $C(K)$-module.

Definition 10 (1,6). Let X be a Banach $C(K)$-module and $x \in X$. The cyclic subspace $X(x)$ of X is defined by

$$X(x) = Cl\{m(a)x : a \in C(K)\},$$

where the notation Cl denotes the closure of a set.

A Banach $C(K)$-module X is said to be a cyclic Banach space if there is an $x \in X$ such that $X = X(x)$.

Similar definition is given on dual Banach space X'. ,

Definition 11. The cyclic subspace $X'(x')$ of X' is defined by

$$X'(x') = Cl\{m^*(a)x' : a \in C(S)\},$$

where the Cl denotes the closure.

The Banach $C(S)$-module X' is called a cyclic Banach space if there exists a vector $x' \in X'$ such that $X' = X'(x')$.

Let X be a Banach $C(K)$-module. Then , it defines a unital bounded homomorphism $m : C(K) \to L(X)$. By means of the Arens product, we define a unital bounded homomorphism $m^* : C(K)'' \to L(X')$. Assume $X' = X'(x_0')$ for some vector $x_0' \in X'$, that is, X' is a cyclic Banach space and x_0' is a cyclic vector. Then, the following assertions are true,[6]:

i. X' can be represented as a Banach lattice with quasi-interior point x_0'.

ii.The cone of X' is identified by the set

$$(X')^+ = Cl\{m^*(a)x' : 0 \leq a \in C(S)\}.$$

iii. The center $Z(X')$ of the Banach lattice X' is the weak* closure of $m^*(C(S))$.

iv. The unit ball of $Z(X')$ is the closure of unit ball of $m^*(S)$ in the weak* operator topology.

v. If x' is quasi-interior point in the Banach lattice X' , then for the order ideal $A_{x'}$ generated by x' we have $A_{x'} = Z(X')x'$.

We give the following theorem concerning cyclic Banach spaces.

Theorem 12. *Assume that X' is a cyclic Banach $C(S)$-module , x_0' is a cyclic vector in X' and let B be the Boolean algebra of all idempotents in $C(S)$. Then,*

$m^*(C(S))$ is weak operator closed and X' has an order continuous norm if it is represented as a Banach lattice.*

Proof. Since the compact space S is Stonian and B is a Bade complete Boolean algebra of projections on X', the conclusion is true.

□

Theorem 13 (6). *Let X be a Banach $C(K)$-module and let $m^* : C(K)'' \to L(X')$ be an injective bounded unital algebra homomorphism such that $m^*(C(S))$ is a weak*operator closed in $L(X')$. The following assertions are equivalent:*

i. X' is a Veksler module and no cyclic subspace of X' contains a copy of l^∞.

ii. Each cyclic subspace of X' has order continuous norm when it is represented as a Banach lattice.

iii. X' is a Kaplansky module and no cyclic subspace of X' contains a copy of l^∞.

Proof. *iii* $\Rightarrow$ *i* . Every Kaplansky module is a Veksler module.

i $\Rightarrow$ *iii*. If X' is represented as a Banach lattice with a quasi-interior point x_0' , then it is σ- Dedekind complete and $m^*(C(S)) = Z(X')$. So, $m^*(C(S))$ is weak* operator closed. Since X' does not contain any copy of ℓ^∞, X' has order continuous norm.

iii $\Rightarrow$ *ii*. Since X' has order continuous norm, X' is Dedekind complete. Hence, it does not contain a copy of ℓ^∞.

□

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