On the analysis of the controlled Chua dynamical system in a cubic modified version

Adela Ionescu, Florian Munteanu

University of Craiova, Romania
Dept of Applied Mathematics
adelajaneta2015@gmail.com

1. Outline. State of art in Chua model behavior

- Nonlinear systems
  - Local behavior
  - Global behavior
- Chaos and Chua’s Circuit
  - Bifurcation
  - Periodic orbits
  - Strange attractors

What is Chaotic System?

- Phenomenon that occurs widely in dynamical systems
- Considered to be complex and no simple analysis
- Study of chaos can be used in real-world applications:

secure communication, medical field, fractal theory, electrical circuits, etc.

What is Chua’s Circuit?

- Autonomous circuit consisting two capacitors, inductor, resistor, and nonlinear resistor.
- Exhibits a variety of chaotic phenomena exhibited by more complex circuits, which makes it popular.
- Readily constructed at low cost using standard electronic components

Nonlinear systems

$$\begin{cases} \dot{X} = F(X); X \in R^n \\ F \in R^n \to R^n \end{cases}$$

- Even for F smooth and bounded for all t ϵ R, the solution X (t)

may become unpredictable or unbounded after some finite time t.

- We divide the study of nonlinear systems into local and global behavior.

Local Behavior

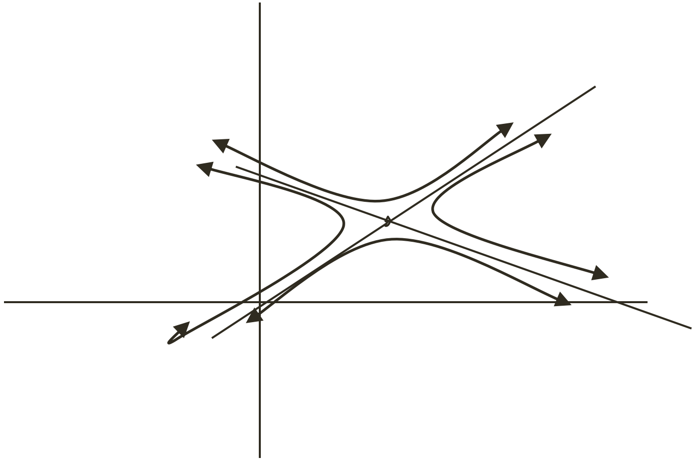
- Idea: use linear systems theory to study nonlinear systems, at least locally, around some special sets, a technique known as *linearization*.

- It is considered:
    - *Linearization around equilibrium points.*
    - *Linearization around periodic* orbits.

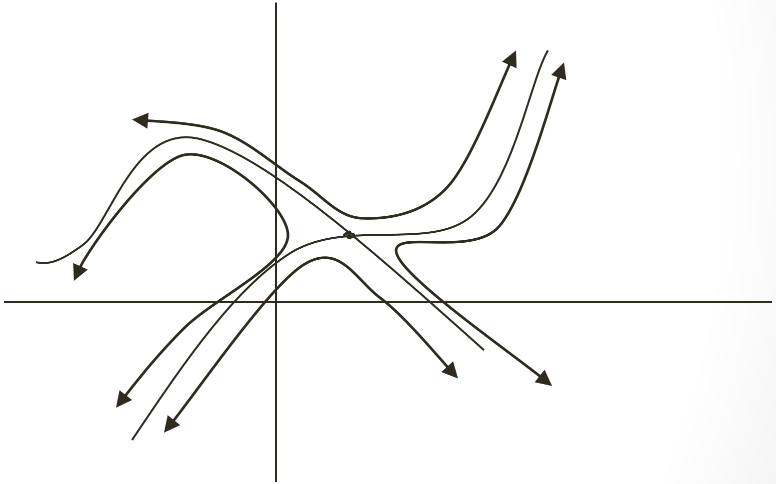
Local Behavior (cont.)

- *Linearization around equilibrium points*
  - Equilibrium point is hyperbolic if no eigenvalues of the Jacobian at the equilibrium point has zero real part.
  - Hartman-Grobman Theorem: nonlinear system has equivalent structure as linearized system, with A=DF(x0), around hyperbolic equilibrium points.

Local behavior (cont.)



Linear system



Non-linear system

Local behavior (cont.)

- *Linearization around periodic orbits*

  - A periodic solution satisfies

$$\begin{cases} \dot{X} = F(X) \\ X(t+\tau) = X(t) \end{cases}, with\ A = DF(X)$$

  - Find periodic orbit by solving the BVP

$$\begin{cases} \dot{X} = F(X) \\ X(0) = X(\tau) \end{cases}$$

  - Determine the Jacobian matrix A(t) = DF(δ)

Local behavior (cont.)

- The *fundamental matrix* of a linear system is the solution of

$$\begin{cases} \dot{\Phi} = A(t)\Phi \\ \Phi(0) = I \end{cases}$$

- If the periodic orbit has period t, then we define the *monodromy matrix* as $\Phi(t)$

- Stability
  - **If |μ|<1, stability**
  - **If |μ|>1, unstability**
  - **If *monodromy* matrix has exactly one eigenvalue with |μ|=1, then the periodic orbit is called *hyperbolic***

Global Behavior

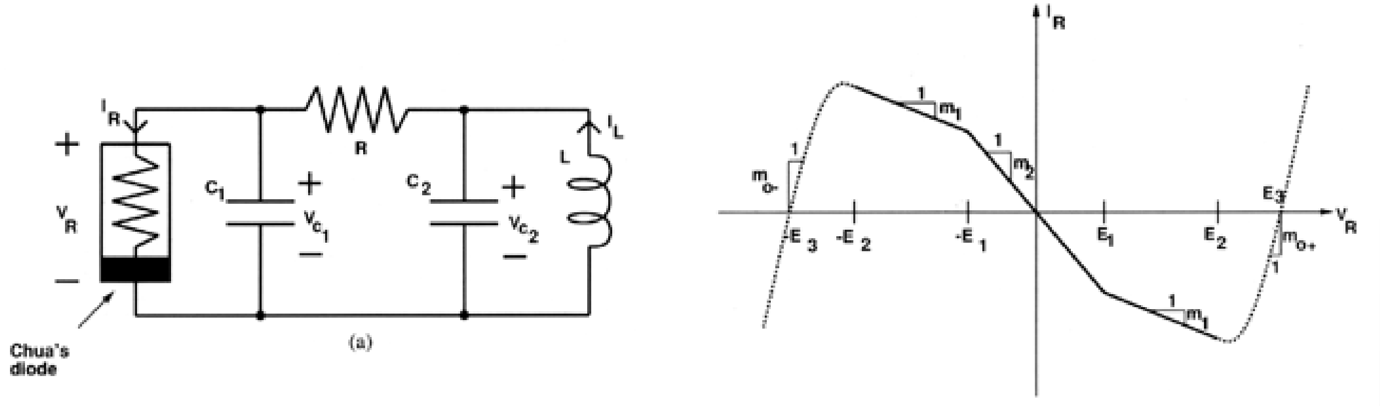
- Study is more complex
- One investigates phenomena such as *heteroclinic* and *homoclinic* trajectories, bifurcations, and chaos.

- we focus in chaos, but this is closely related to the other concepts and phenomena mentioned above.

Chaos and Chua's Circuit

➢Main goal is to give brief introduction to underlying ideas behind the notion of chaos, by studying the system that models Chua's circuit.

➢Chua's circuit consists of two capacitors C1, C2, one inductor L, one resistor R, and one non-linear resistor (Chua's diode).



(a) Chua's Circuit. (b) Driving-point characteristic of Chua's Diode.

Chua's Circuit (cont.)

If we let X1 = V1, X2 = V2 and X3 = I3, Chua's circuit is

$$\begin{cases} \dot{X}_1 = \alpha[X_2 - h(x)] \\ \dot{X}_2 = X_1 - X_2 + X_3 \\ \dot{X}_3 = -\beta X_2 \end{cases}$$

$$where\, \beta = 14.3$$

$$h(x) = \frac{2}{7}x - \frac{3}{14}\left(|x+1| - |x-1|\right)$$

$$|x| \approx \frac{2}{\pi}\arctan(10x)$$

Chua’s Circuit (cont.)

If we let X1 = V1, X2 = V2 and X3 = I3, the Chua's circuit is

$$\begin{cases} X_1 = \alpha[X_2 - h(x)] \\ X_2 = X_1 - X_2 + X_3 \\ X_3 = -\beta X_2 \end{cases}$$

$$where\, \beta = 14.3$$

$$h(x) = \frac{2}{7}x - \frac{3}{14}\left(|x+1| - |x-1|\right)$$

$$|x| \approx \frac{2}{\pi}\arctan(10x)$$

The Jacobian matrix is $J(x) = \begin{bmatrix} \alpha h^{'}(x_1) & \alpha & 1 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}$

where $h^{'}(x_1) = a_1 + \frac{10}{\pi}(a_0 - a_1)\left[\frac{1}{1+100(x+1)^2} - \frac{1}{1+100(x-1)^2}\right]$

Chua's circuit (cont.)

- At (0,0,0) we have

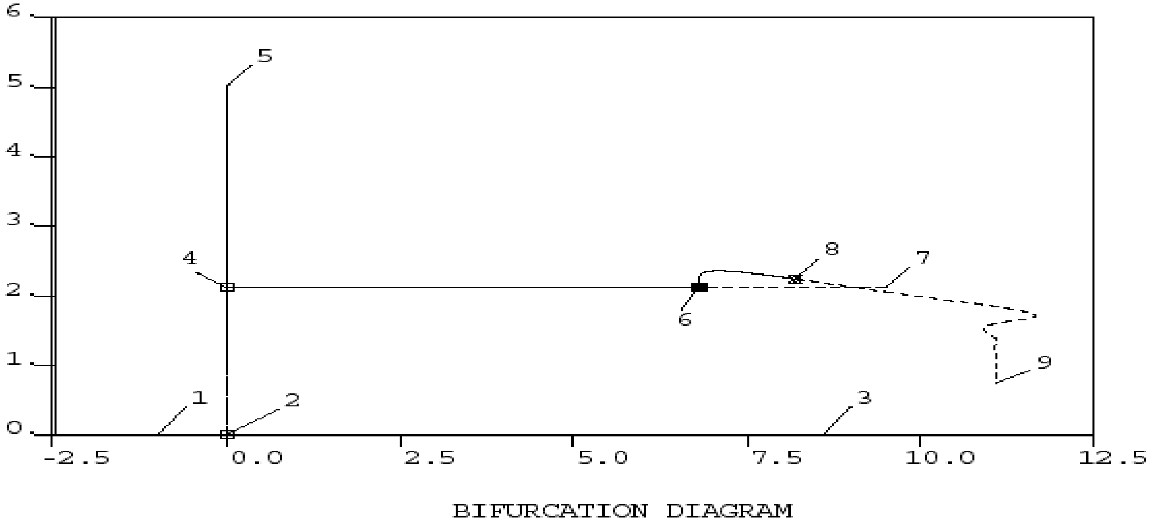
$$J_F\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{bmatrix}0 & \alpha & 0\\ 1 & -1 & 1\\ 0 & -\beta & 0\end{bmatrix}$$

- Eigenvalues are

$$\lambda=\frac{-1\pm\sqrt{1+4(\beta-\alpha)}}{2}$$

Bifurcation

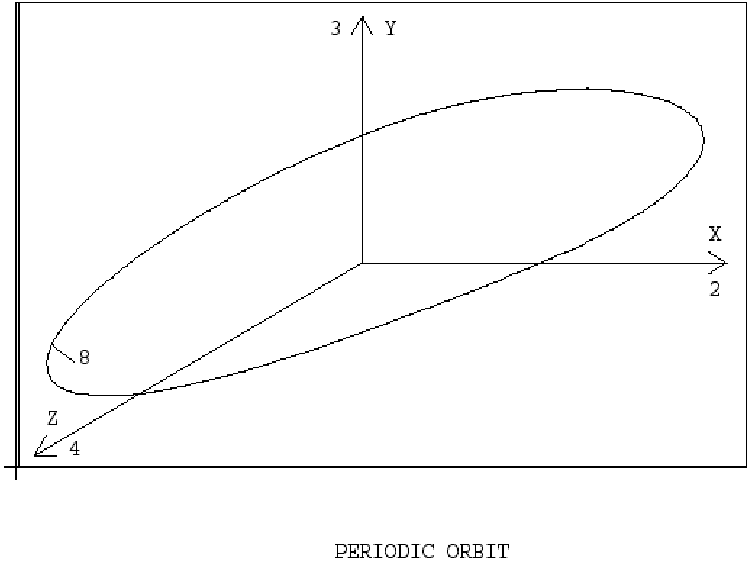
- Bifurcation diagram starting value α = -1



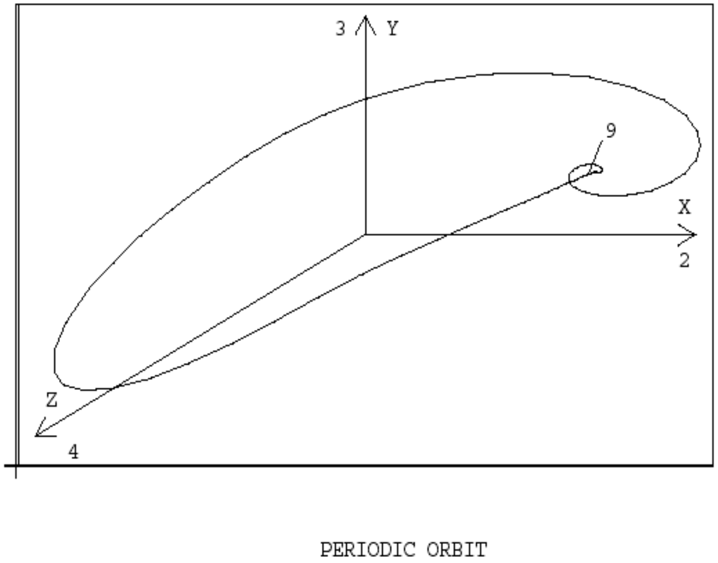
- Plot shows norm of the solution ||x|| versus parameter α.

Periodic orbits

➢Following Hopf bifurcation, two periodic orbits appear. The first with period 2.2835 (for α =8.19613) and the second with period 19.3835 (for α=11.07941)



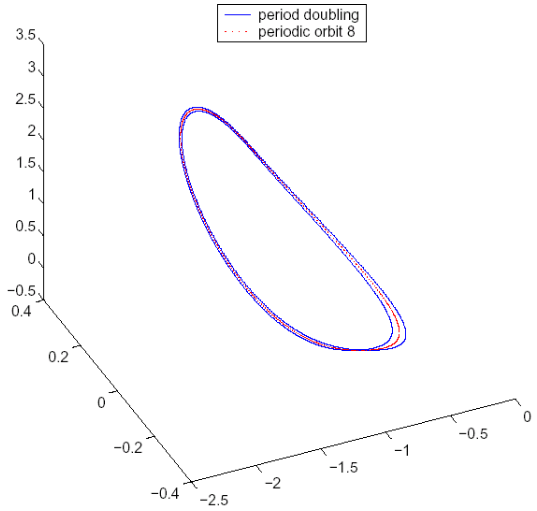
1st periodic orbit



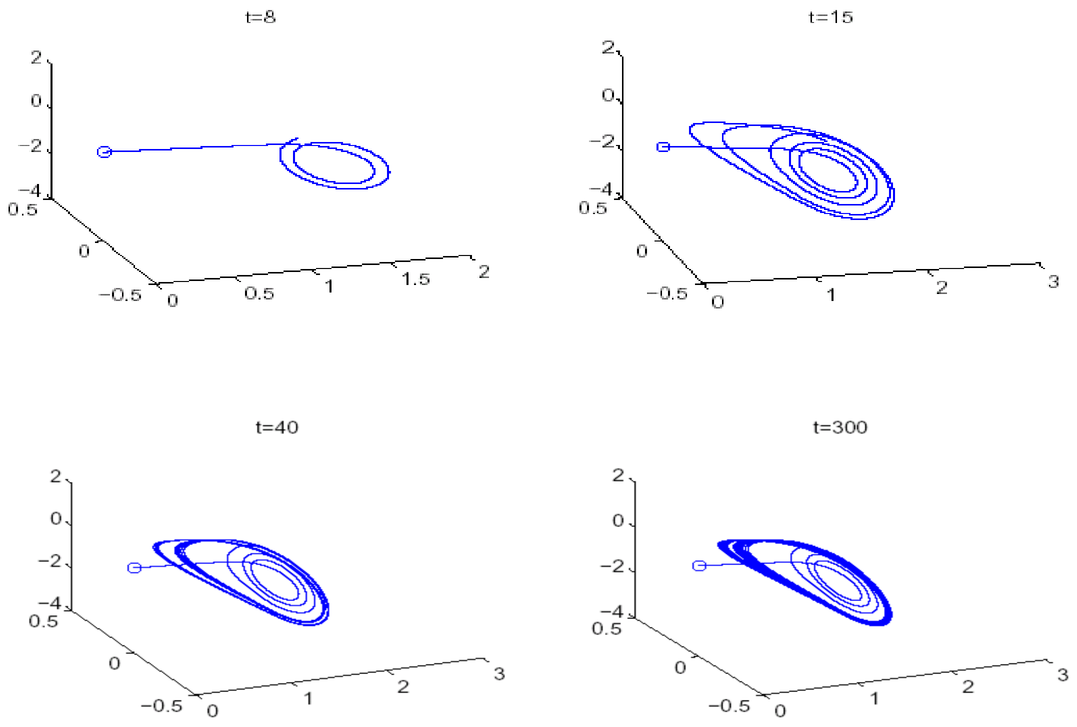
2nd periodic orbit

Sensitivity to initial data

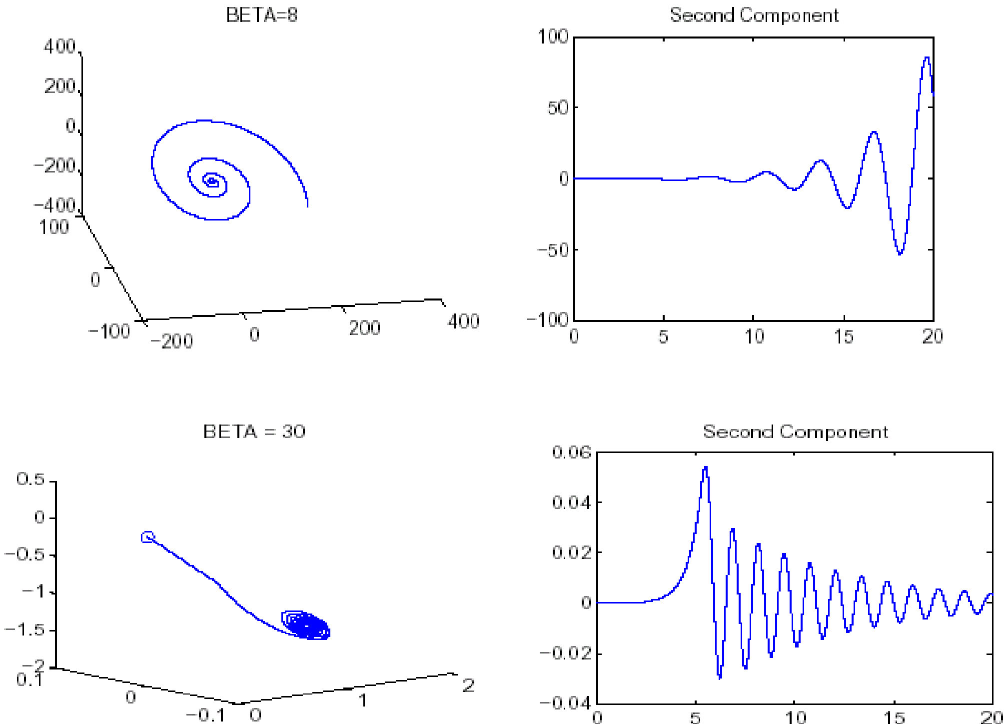
To show that this dynamical system is sensitive to small changes in the data (one sign of the presence of chaos), we solve the system again for α=8.196 (not=8.196013). However, we obtain a different periodic orbit, which seems to "encircle" the previous one.



Strange attractors

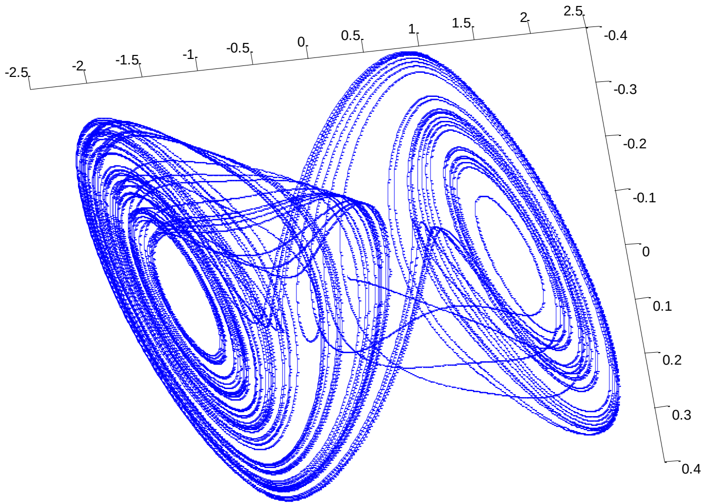


Strange attractors (cont.)



Strange attractor (cont.)

Finally, we compute another strange attractor solution to Chua’s circuit, which is known in literature as ***double-scroll attractor***. This type of attractor has been mistaken for experimental noise, but they are now commonly found in digital filter and synchronization circuits.



2. Recent statements for Chua model
2.1. Feedback linearization for a dynamical system

- The term *feedback* is used to refer to a situation in which two (or more) dynamical systems are connected together such that each system influences the other and their dynamics are thus strongly coupled.

- Feedback is a powerful idea which is used extensively in natural and technological systems. The principle of feedback is simple: *base correcting actions on the difference between desired and actual performance*.

- The use of feedback has often resulted in vast improvements in system capability and these improvements have sometimes been revolutionary. The reason for this is that feedback has some truly remarkable properties

Feedback linearization- two basic properties

- One of the key uses of feedback is *to provide robustness to uncertainty*. By measuring the difference between the sensed value of a regulated signal and its desired value, we can supply a corrective action.

- Another use of feedback is to *change the dynamics* of a system. Through feedback, we can alter the behavior of a system to meet the needs of an application: systems that are unstable can be stabilized, systems that are sluggish can be made responsive and systems that have drifting operating points can be held constant.

Feedback linearization-enhanced by **control**

- Control theory provides a rich collection of techniques to analyze the *stability* and *dynamic response* of complex systems and to place bounds on the behavior of such systems by analyzing the gains of linear and nonlinear operators that describe their components

- A modern controller senses the operation of a system, compares that against the desired behavior, computes corrective actions based on a model of the system’s response to external inputs and actuates the system to effect the desired change. This basic *feedback loop* of sensing, computation and actuation is the central concept in control

2.2. Results for Chua model in a slightly modified version

- A recent aim – to control the Chua dynamical system to a stable state. The original form of the model was taken into account
- (1) $$\begin{cases} \dot{x} &= \alpha(y - h(x)) \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y \end{cases}$$
- with α and β the bifurcation parameters.
- A widely used form for the piece-wise function $h(x)$ is the following
- (2) $$h(x) = \begin{cases} m_1(x+1) - m_0 & , \quad x < -1 \\ m_{0x} & , \quad -1 \le x \le 1 \\ m_1(x-1) + m_0 & , \quad x > 1 \end{cases}$$

First case of controlling the Chua model to a stable state

- The following dynamical system was considered
- (3) $$\begin{cases} \dot{x} = \alpha(y - m_0 x) \\ \dot{y} = x - y + z + kx, \qquad -1 \le x \le 1, k \in R \\ \dot{z} = -\beta y \end{cases}$$
- With a control *u*(*x*) = *kx*, k real, on the second component of the system
- The equilibrium states and the Jacobian were calculated, and in order to test the eigenvalues, the **Routh- Hurwicz criterion** was taken into account:
- A third order polynomial $s^3 + a_2 s^2 + a_1 s + a_0$
- has all roots in the left open-half plane if and only if
- (4) $a_0, a_2 > 0 \;\; and \; a_2 \cdot a_1 > a_0$
- Note "left open –half plane" $\leftrightarrow Re(\lambda) < 0 \leftrightarrow$ stability

First case of controlling the Chua model to a stable state

- For the system (3) from the (R-H) conditions, the **feasible** but strong relationships between the parameters were obtained:
- (5)

$$\begin{cases} m_0\alpha + 1 < 0, \qquad m_0\alpha\beta < 0 \\ -(1 + m_0\alpha)[\alpha(1 + k) - m_0\alpha - \beta] > -m_0\alpha\beta \end{cases}$$

- And further
- (6.1) $\beta > (m_0\alpha - \alpha - \alpha k)(1 + m_0\alpha)$
- Which could be managed !!

Second case of controlling the Chua model to a stable state

- A slightly modified version of the model was taken into account, with a control on the third component of the dynamical system
- (7) $$\begin{cases} \dot{x} = \alpha(y - m_0 x) \\ \dot{y} = x - y + z \\ \dot{z} = -\beta y + kx \end{cases}, \qquad -1 \le x \le 1, k \in R$$
- The origin is also a equilibrium state in this case, too.
- Re-taking the same way – with (R-H) criterion, the following relationship between the parameters was obtained:
- (6.2) $$\beta > \alpha[(1 - m_0)(1 + \alpha m_0) - k]$$
- Slightly different comparing with (6.1), but feasible too!!

3. For the present paper – the **cubic** version of the Chua model

- The cubic modified Chua circuit system – a recent challenge which takes into account a cubic polynomial form for the function h(x). That is
- (8) $$\begin{cases} \dot{u} &= \alpha\left(v - au - bu^2 - cu^3\right) \\ \dot{v} &= u - v + w \\ \dot{w} &= -\beta v - \gamma w \end{cases}$$
- where $\dot{u} = \frac{du}{dt}$, $\dot{v} = \frac{dv}{dt}$ and $\dot{w} = \frac{dw}{dt}$ and $\alpha = \frac{C_2}{C_1}$, $\beta = \frac{C_2 R^2}{L}$ and $\gamma = \frac{C_2 R R_0}{L}$.
- $a = G_a R - 1, b = G_b R, c = mR, \quad m \in R$ and G_a, G_b are the slopes of the characteristic in the inner and outer regions, respectively

- M. K. Gupta and C. K. Yadav, Jacobi stability of modied Chua circuit system, Int. J. Geom. Meth. Mod. Phys., vol 14, no 6, 1750089 (21pages), 2017

For the **cubic** version of the Chua model

- Two approaches are very recent and challenging:
- The KCC (Kosambi-Cartan-Chern) theory –the basic idea is that the second order differential equations (SODEs) which models the dynamical system and geodesic equations in associated Finsler space are topologically equivalent

- The Jacobi stability is a natural generalization of the stability of the geodesic flow on a differentiable manifold endowed with a metric (Riemannian or Finslerian) to the non-metric setting (based on the deviation curvature tensor)
- For the cubic Chua model, the Jacobi stability was approached with good results:

- F. Munteanu, A. Ionescu, "Analysing the nonlinear dynamics of a cubic modified Chua's circuit system", 2021 International Conference on Applied and Theoretical Electricity (ICATE), DOI: 10.1109/ICATE49685.2021.9465025

Controlling the Chua model to a stable state, in the cubic version

- A simple controller is considered on the second component of the model, namely
- (9) $$\begin{cases} \dot{u} = \alpha\left(v - au - bu^2 - cu^3\right) \\ \dot{v} = u - v + w + kv \\ \dot{w} = -\beta v - \gamma w \end{cases}$$
- The system has three equilibriums → more complex → focus the analysis for the moment, only on the origin: $E_1 = (0,0,0)$
- The aim – to check if we can control the system as in (9) to a stable state
- The way – R-H criterion again

Controlling the Chua model to a stable state, in the cubic version

- Getting the way of R-H criterion, for the zero equilibrium E1 (Jacobean, characteristic polynomial) → the following coefficients were obtained for the 3rd order polynomial:

- (10)

$$a_0 = \alpha(a\beta + ak\gamma - a\gamma + \gamma)$$
$$a_1 = \alpha ak - \alpha a\gamma - \alpha a + \gamma k - \gamma + \alpha + \beta$$
$$a_2 = k - \gamma - \alpha a - 1$$

- And thus the conditions (4) become

- (11)

$$\alpha(a\beta + ak\gamma - a\gamma + \gamma) > 0, k - \gamma - \alpha a - 1 > 0$$
$$(k - \gamma - \alpha a - 1)(\alpha ak - \alpha a\gamma - \alpha a + \gamma k - \gamma + \alpha + \beta) > \alpha(a\beta + ak\gamma - a\gamma + \gamma)$$

- Much more difficult to manage, comparing to rel. (6) → work in progress !!

Remarks. Aims

- Signs of chaos:
  - Sensitivity to initial data
  - Strange attractors
  - Unpredictability
- If chaos can be *understood* with elementary knowledge of linear algebra and differential equations, it can be better *approached and analyzed by control theory* the above calculus will be enhanced with **graphical** comparative analysis (work in progress !!)
- **The Chua dynamical system can be controlled up to a stable state.** In the cubic case, managing the calculus is more complex → testing different controls will be useful

- Also considering different options for the function h will be helpful

References (selected)

[1] W. E. Boyce, R.C. DiPrima, *Elementary Differential Equations*, seventh edition, John Wiley & Sons, Inc. (2003)

[2] L. Dieci and J. Rebaza, “ *Point to point and point to periodic connections*”, BIT, Numerical Mathematics, 2004.

[3] E. Doedel, A. Champneys, T. Fairgrieve, Y. Kuznetsov, B. Sandstede, and X. Wang. AUTO 2000: Continuation and bifurcation software for ordinary differential equations. (2000). ftp://ftp.cs.concordia.ca.

[4] J. Hale and H. Kocak, *Dynamics and Bifurcations*, third edition, Springer Verlag (1996).

[5] M. P. Kennedy, “*Three steps to chaos, I: Evolution*”, IEEE Transactions on circuits and Systems, Vol. 40, No 10 (1993)

[6] M. P. Kennedy, “*Three steps to chaos, II: A Chua’s circuit primer*”, IEEE Transactions on circuits and Systems, Vol. 40, No 10 (1993)

[7] L. Torres and L. Aguirre, “ *Inductorless Chua’s circuit*”, Electronic letters, Vol. 36, No 23 (2000)

References (selected)

- [8] Andrievskii, B.R., Fradkov, A.L., *Control of chaos: methods and applications, II Applications,* Automation and Remote control 65 (4) 2004

- [9] Ott, E., Grebogi, C., Yorke, J.A., *Controlling chaos*, Phys. Rev. Lett 64 (1999), 1179-1184

- [10] Efrem, R, Ionescu, A., Munteanu, F., "*On a recursive method for feedback linearization of nonlinear systems. The case of mixing flow model*", Atti della Accademia Peloritana dei Pericolanti, vol 97, No S2 (2019)

- [11] Ionescu, A., Munteanu, F., "*A geometrical and computational analysis of Chua's circuit system*". 2016 International Conference on Applied and Theoretical Electricity (ICATE), IEEE Xplore, pp 1-5, http://ieeexplore.ieee.org/document/7754602

THANKS FOR YOUR ATTENTION!