FINITE CARTESIAN PRODUCT OF CHAIN CONNECTED SETS IN TOPOLOGICAL SPACES

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Introduction

Let X be a topological space and $\mathcal{U}$ be an open cover of X.

Two points $x, y \in X$ are said to be $\mathcal{U}$-chain connected in X if there exist $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $x \in U_1,\ y \in U_n$ and $U_i \cap U_{i+1} \neq \emptyset, \forall i \in \{1, 2, \ldots, n-1\}$.

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If two points $x, y \in X$ are $\mathcal{U}$-chain connected in X, for any open cover $\mathcal{U}$ of X, then we say that x and y are chain connected in X.

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The relations of $\mathcal{U}$-chain connectedness and chain connectedness are equivalence relations on X.

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If $C \subseteq Y \subseteq X$ is chain connected in Y, then it is chain connected in X.

Corollary (4)

Every connected set is chain connected in each of its superspaces. The converse claim does not hold in general.

Introduction

Example (5)

Consider $C = \{(0,0),(1,0)\}$ and $X = C \cup \bigcup_{n\in\mathbb{N}} \left([0,1] \times \frac{1}{n}\right)$. Then C is chain connected in X, but not connected.

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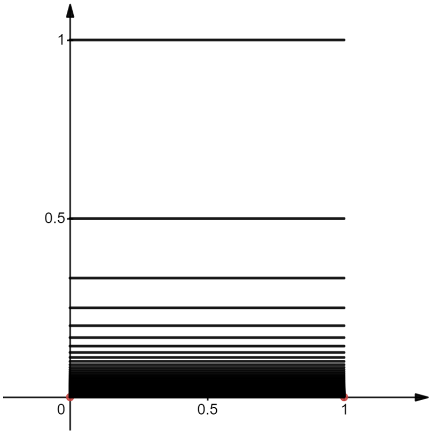

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Corollary (7)

If $f : X \to Y$ is a homeomorphism, a set C is chain connected in X if and only if $f(C)$ is chain connected in Y.

Main results

Theorem (8)

If $A_1, A_2, \ldots, A_n$ are chain connected sets in $X_1, X_2, \ldots, X_n$ respectively, then $\Pi_{i=1}^{n} A_i$ is a chain connected set in $\Pi_{i=1}^{n} X_i$.

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Proof: We use mathematical induction to prove the theorem. First, we consider the case when $n = 2$.

Let A_X and A_Y be chain connected sets in X and Y respectively and let $\mathcal{U}$ be a covering of $X \times Y$. If $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are canonical projections, then $\mathcal{U}_X = \{\pi_X(U) | U \in \mathcal{U}\}$ and $\mathcal{U}_Y = \{\pi_Y(U) | U \in \mathcal{U}\}$ are coverings of X and Y respectively.

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Proof: (continues) Let $(x_1, y_1), (x_2, y_2) \in A_x \times A_Y$. Since A_X and A_Y are chain connected sets in X and Y respectively, there exist a chain $U_1^X, U_2^X, \ldots, U_{m_X}^X$ in $\mathcal{U}_X$ from x_1 to x_2 and a chain $U_1^Y, U_2^Y, \ldots, U_{m_Y}^Y$ in $\mathcal{U}_Y$ from y_1 to y_2. Then $U_1^X \times U_1^Y, U_1^X \times U_2^Y, \ldots, U_1^X \times U_{m_Y}^Y, U_2^X \times U_{m_Y}^Y, U_3^X \times U_{m_Y}^Y, \ldots, U_{m_X}^X \times U_{m_Y}^Y$, is a chain in $\mathcal{U}$ from (x_1, y_1) to (x_2, y_2). Hence $A_X \times A_Y$ is a chain connected set in $X \times Y$.

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The same technique is used to prove the general case.

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Corollary (9)

If $X_1, X_2, \ldots, X_n$ are connected spaces, then $\Pi_{i=1}^n X_i$ with the product topology, is a connected space.

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If A is a chain connected set in X and B is a connected set, then $A \times B$ is a chain connected set in $X \times B$.

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Example (11)

Consider $C = \{(0,0),(1,0)\}$ and $X = C \cup \bigcup_{n \in \mathbb{N}} \left([0,1] \times \frac{1}{n}\right)$ as in Example 5. Then $C \times [0,1]$ is a chain connected set in $X \times [0,1]$ but not connected.

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Conclusion

Theorem 8 proves that the finite Cartesian product of chain connected sets in respective spaces is a chain connected set in the product space.

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Example 11 shows that the product of chain connected set in a space with a connected set need not be connected.

Thank you for the attention!