Homogenization of a Diffusion Problem in Thin Filtering Materials

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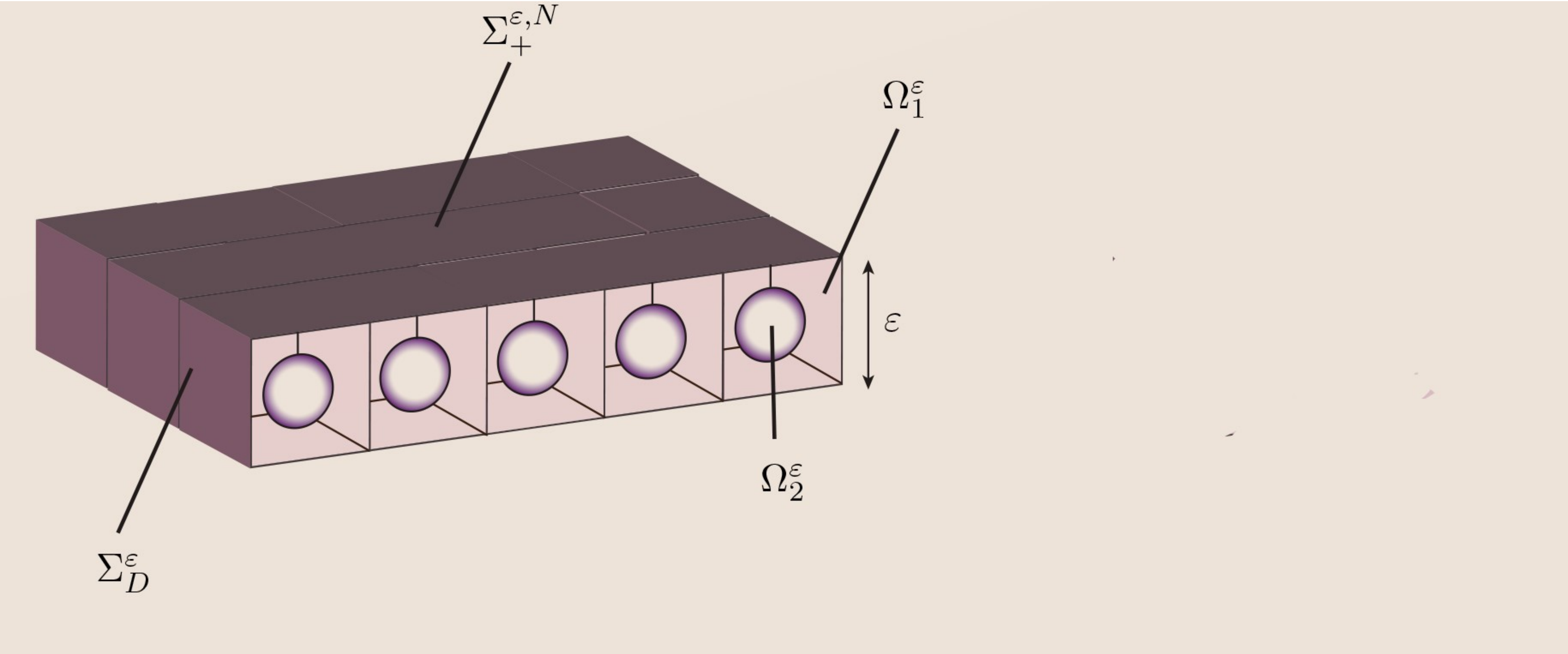
- Motivation: to obtain homogenization results for a diffusion problem in a thin periodic heterogeneous composite medium $\Omega^\varepsilon = \omega \times (0, \varepsilon)$ of small height ε, made up of two materials separated by an imperfect interface.

- The two components of the periodic domain Ω^ε, namely Ω_1^ε, supposed to be connected, and, respectively, Ω_2^ε, assumed to be disconnected, are separated by an imperfect interface Γ^ε. Such a structure might be encountered in applications to problems involving filtering materials constituted of three thin horizontal layers of total height ε, as for instance textiles, paper, or biological tissues.

- Main features: the special geometry (small height, periodicity, connectivity), the presence of two materials and the discontinuities of the solution and of its flux across the interface separating the two materials.


$\Sigma_{+}^{\varepsilon,N}$
Ω_1^{ε}
ε
Ω_2^{ε}
Σ_D^{ε}

The microscopic problem

Let ω be a smooth and bounded domain in $\mathbb{R}^2$. The independent variable $x \in \mathbb{R}^3$ is denoted by $x = (x_1, x_2, x_3) = (\bar{x}, x_3)$. We define

$$\Omega^\varepsilon = \omega \times (0, \varepsilon) = \{x = (\bar{x}, x_3) \in \mathbb{R}^3 | \, \bar{x} \in \omega, \, 0 < x_3 < \varepsilon\}.$$

- **Goal**: to describe the asymptotic behavior, as $\varepsilon \to 0$, of the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the following problem:

$$\begin{cases} -\mathrm{div}\,(A^\varepsilon \nabla u_1^\varepsilon) = f \quad \text{in } \Omega_1^\varepsilon, \\ -\mathrm{div}\,(\varepsilon^\beta A^\varepsilon \nabla u_2^\varepsilon) = f \quad \text{in } \Omega_2^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot n^\varepsilon = \varepsilon h^\varepsilon \,(u_1^\varepsilon - u_2^\varepsilon) - G^\varepsilon \quad \text{on } \Gamma^\varepsilon, \\ \varepsilon^\beta A^\varepsilon \nabla u_2^\varepsilon \cdot n^\varepsilon = \varepsilon h^\varepsilon \,(u_1^\varepsilon - u_2^\varepsilon) \quad \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot \nu_\pm^\varepsilon = \varepsilon k_\pm \quad \text{on } \Sigma_\pm^{\varepsilon,N}, \\ u_1^\varepsilon = 0 \quad \text{on } \Sigma_D^\varepsilon. \end{cases}$$

Assumptions

(A1) Let $\lambda, \mu \in \mathbb{R}$, with $0 < \lambda \leq \mu$ and denote by $\mathcal{M}(\lambda, \mu, Y)$ the set of all the matrices $A = (a_{ij}) \in (L^\infty(Y))^{3\times 3}$ such that for any $\xi \in \mathbb{R}^3$, $\lambda|\xi|^2 \leq (A(y)\xi,\, \xi) \leq \mu|\xi|^2$, a.e. in Y. Let $A \in \mathcal{M}(\lambda, \mu, Y)$ be symmetric and 1–periodic in its first two variables y_1 and y_2. We put

$$A^\varepsilon(x) = A^\varepsilon(\bar{x}, x_3) = A\left(\frac{\bar{x}}{\varepsilon}, \frac{x_3}{\varepsilon}\right) = A\left(\frac{x}{\varepsilon}\right) \text{ a.e. in } \Omega^\varepsilon.$$

(A2) The functions $f \in L^2(\omega)$, $k_+ \in L^2(\omega)$ and $k_- \in L^2(\omega)$ are given.

(A3) Let h be a function 1–periodic in the first two variables y_1 and y_2 such that $h \in L^\infty(\Gamma)$ and there exists $h_0 \in \mathbb{R}$ with $0 < h_0 < h(y)$ a.e. on Γ. We set

$$h^\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right) \qquad \text{a.e. on } \Gamma^\varepsilon.$$

(A4) g is 1–periodic in the first two variables y_1 and y_2 and belongs to $L^2(\Gamma)$. We define

$$g^\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right) \text{ a.e. on } \Gamma^\varepsilon.$$

We take (i) $G^\varepsilon = g^\varepsilon$ and $\mathcal{M}_\Gamma(g) = 0$ OR (ii) $G^\varepsilon = \varepsilon g^\varepsilon$ and $\mathcal{M}_\Gamma(g) \neq 0$.

Previous results

Classical porous media (i.e. **not of thin height**).

$G^\varepsilon = 0$ (no jump in the flux): J. L. Auriault, H. Ene (1994); H. Ene, D. Polisevski (2002); S. Monsurrò (2003); S. Monsurrò, P. Donato (2004); R.B., D. Polisevski (2004, 2005); C. Timofte (2010, 2013, 2014); P. Donato, K.H. Le Nguyen, R. Tardieu (2011); G. Allaire, Z. Habibi (2013); M. Amar, D. Andreucci, R. Gianni, C. Timofte (2020).

$G^\varepsilon \neq 0$ (jump in the flux): K. Fellner, V. Kovtunenko (2015); R. B., C. Timofte (2016, 2017, 2018, 2019).

Thin porous media (geometry as defined here) and jump in flux.

E. R. Ijioma, A. Muntean, T. Ogawa (2015) (no jump in the solution)

$\beta = 2$ and $G^\varepsilon = g^\varepsilon$ or $G^\varepsilon = \varepsilon g^\varepsilon$; R. B., C. Timofte (2020).

Thin porous media between two layers of fixed height.

M. Neuss-Radu, W. Jäger (2007), M. Gahn, P. Knabner, M. Neuss-Radu (2016); V. Raveendran, E. N.M. Cirillo, I. de Bonis, A. Muntean (2021).

Case $\beta = 0$ **and** $G^\varepsilon = \varepsilon g^\varepsilon$**.**

- variational formulation of the problem and well-posedness
- derivation of *a priori* estimates for the solution of the problem
- compactess results and convergence
- passage to the limit $\varepsilon \to 0$ in the variational formulation and derivation of the limit problem with two scales
- derivation of the homogenized problem and its analysis

The variational formulation

- Consider the Hilbert space $H^\varepsilon = V^\varepsilon \times H^1(\Omega_2^\varepsilon)$, with $V^\varepsilon = \left\{ v \in H^1(\Omega_1^\varepsilon) |\, v = 0 \text{ on } \Sigma_D^\varepsilon \right\}$,

endowed with the norm

$$\|v\|_{H^\varepsilon}^2 = \|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}^2 + \|\nabla v_2\|_{L^2(\Omega_2^\varepsilon)}^2 + \varepsilon \|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)}^2.$$

- The variational formulation of the problem is : find $u^\varepsilon \in H^\varepsilon$ s.t.

$$a(u^\varepsilon, v) = l(v), \quad \forall v \in H^\varepsilon,$$

where the bilinear form $a : H^\varepsilon \times H^\varepsilon \to \mathbb{R}$ and the linear form $l : H^\varepsilon \to \mathbb{R}$ are

$$a(u,v) = \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1 \nabla v_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2 \nabla v_2 \,\mathrm{d}x + \varepsilon \int_{\Gamma^\varepsilon} h^\varepsilon (u_1 - u_2)(v_1 - v_2) \,\mathrm{d}\sigma_x,$$

$$l(v) = \int_{\Omega_1^\varepsilon} f v_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} f v_2 \,\mathrm{d}x + \varepsilon \int_{\Gamma^\varepsilon} g^\varepsilon v_1 \,\mathrm{d}\sigma_x +$$

$$\varepsilon \int_{\Sigma_+^{\varepsilon,N}} k_+ v_1 \,\mathrm{d}\sigma_x^+ + \varepsilon \int_{\Sigma_-^{\varepsilon,N}} k_- v_1 \,\mathrm{d}\sigma_x^- .$$

- **Theorem**. For any $\varepsilon \in (0,1)$, the variational problem has a unique solution $u^\varepsilon \in H^\varepsilon$. Moreover, there exists a constant $C > 0$, independent of ε, such that

$$\frac{1}{\sqrt{\varepsilon}}\|u_\alpha^\varepsilon\|_{L^2(\Omega_\alpha^\varepsilon)} \le C, \quad \frac{1}{\sqrt{\varepsilon}}\|\nabla u_\alpha^\varepsilon\|_{L^2(\Omega_\alpha^\varepsilon)} \le C, \quad \alpha \in \{1,2\},$$

$$\|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \le C.$$

- Convergence results are derived by the periodic unfolding method adapted to our thin geometry. This method allows us to simultaneously perform homogenization and dimension reduction (see D. Onofrei (2006); M. Neus-Radu, W. Jäger (2007); G. Griso, A. Migunova, J. Orlik (2017); D. Cioră nescu, A. Damlamian, G. Griso (2018)).

We use suitable unfolding operators $\mathcal{T}_\alpha^\varepsilon$ ($\alpha = 1,2$), mapping functions defined on the oscillating domains $\Omega_\alpha^\varepsilon$ into functions defined on the fixed domains $\omega \times Y_\alpha$.

- For $x \in \mathbb{R}^3$, we have $x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right)$. For $x \in \Omega^\varepsilon$, we have

$$x = \varepsilon\left(\left[\frac{(\bar{x},0)}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right).$$

- For any Lebesgue measurable function φ on $\Omega_\alpha^\varepsilon$, $\alpha \in \{1,2\}$, we define the periodic unfolding operators by the formula

$$\mathcal{T}_\alpha^\varepsilon(\varphi)(\bar{x}, y) = \varphi\left(\varepsilon\left[\frac{(\bar{x},0)}{\varepsilon}\right]_Y + \varepsilon y\right), \quad \text{for a.e. } (\bar{x}, y) \in \omega \times Y_\alpha.$$

- For any function φ which is Lebesgue-measurable on Γ^ε, the periodic boundary unfolding operator $\mathcal{T}_b^\varepsilon$ is defined by

$$\mathcal{T}_b^\varepsilon(\varphi)(\bar{x}, y) = \varphi\left(\varepsilon\left[\frac{(\bar{x},0)}{\varepsilon}\right]_Y + \varepsilon y\right), \quad \text{for a.e. } (\bar{x}, y) \in \omega \times \Gamma.$$

- We have two different scales: the macroscopic scale $\bar{x}$, giving the position of a point in the domain ω and the microscopic scale y, which gives the position of a point in Y.
- The function $\mathcal{T}_\alpha^\varepsilon(\varphi)$ is Lebesgue-measurable on $\omega \times Y_\alpha$ (we increased the number of variables). The oscillations of the function φ are put in the variable y (separation of the scales).

$$\lim_{\varepsilon \to 0} \int_{\Omega_\alpha^\varepsilon} \varphi(x)\,\mathrm{d}x = \lim_{\varepsilon \to 0} \varepsilon \int_{\omega \times Y_\alpha} \mathcal{T}_\alpha^\varepsilon(\varphi)(\bar{x}, y)\,\mathrm{d}\bar{x}\,\mathrm{d}y.$$

Homogenization results

- Goal: to pass to the limit, with $\varepsilon \to 0$, in the variational formulation.
- Technique: the periodic unfolding method and general compactness results.
- There exist $u_1 \in H_0^1(\omega)$, $u_2 \in L^2(\omega)$, $\widehat{u}_1 \in L^2(\omega, H^1_{\text{per}}(Y_1))$, $\widehat{u}_2 \in L^2(\omega, H^1(Y_2))$ with $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$, $\mathcal{M}_\Gamma(\widehat{u}_2) = 0$ and such that, up to a subsequence, for $\varepsilon \to 0$, we get:

$$
\begin{aligned}
&\mathcal{T}_1^\varepsilon(u_1^\varepsilon) \rightharpoonup u_1 \quad \text{weakly in } L^2(\omega, H^1(Y_1)),\\
&\mathcal{T}_1^\varepsilon(\nabla_{\bar{x}} u_1^\varepsilon) \rightharpoonup \nabla_{\bar{x}}\, u_1 + \nabla_{\bar{y}} \widehat{u}_1 \quad \text{weakly in } L^2(\omega \times Y_1),\\
&\mathcal{T}_1^\varepsilon(\partial_{x_3} u_1^\varepsilon) \rightharpoonup \partial_{y_3} \widehat{u}_1 \quad \text{weakly in } L^2(\omega \times Y_1),\\
&\mathcal{T}_2^\varepsilon(u_2^\varepsilon) \rightharpoonup u_2 \quad \text{weakly in } L^2(\omega, H^1(Y_2)),\\
&\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla_y \widehat{u}_2 \quad \text{weakly in } L^2(\omega \times Y_2).
\end{aligned}
$$

- Notation: to every $w = w(\bar{x}) \in H^1(\omega)$, whose gradient $\nabla_{\bar{x}} w(\bar{x})$ has two components, we associate the tridimensional vector $\overline{\nabla} w(\bar{x})$ defined by

$$\overline{\nabla} w(\bar{x}) = (\nabla_{\bar{x}} w(\bar{x}), 0).$$

• **Theorem.** Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be the unique solution of the variational problem. Then, the limit functions $u_1 \in H_0^1(\omega)$, $u_2 \in L^2(\omega)$, $\widehat{u}_1 \in L^2(\omega, H^1_{\text{per}}(Y_1))$, and $\widehat{u}_2 \in L^2(\omega, H^1(Y_2))$ are such that $\nabla_{\bar{y}} \widehat{u}_2 = 0$, the pair $(u_1, \widehat{u}_1)$ is the unique solution of the two-scales limit problem

$$\int_{\omega\times Y_1} A(y)(\overline{\nabla} u_1 + \nabla_y \widehat{u}_1)(\overline{\nabla}\varphi + \nabla_y \Phi)\,\mathrm{d}\bar{x}\,\mathrm{d}y = \int_\omega f(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x} +$$

$$|\Gamma|\mathcal{M}_\Gamma(g)\int_\omega \varphi(\bar{x})\,\mathrm{d}\bar{x} + \int_\omega k_+(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x} + \int_\omega k_-(\bar{x})\varphi(\bar{x})\,\mathrm{d}\bar{x},$$

for all $\varphi \in H_0^1(\omega)$ and $\Phi \in L^2(\omega, H^1_{\text{per}}(Y_1))$, and

$$u_2(\bar{x}) = u_1(\bar{x}) + \frac{|Y_2|}{|\Gamma|\mathcal{M}_\Gamma(h)} f(\bar{x}).$$

- **Theorem.** Let $(u_1, \widehat{u}_1)$ be the unique solution of the two-scales limit problem. Then, u_1 satisfies the following homogenized problem

$$\begin{cases} -\operatorname{div}_{\bar{x}} \left(A^{\text{hom}} \nabla_{\bar{x}} u_1(\bar{x})\right) = f(\bar{x}) + |\Gamma| \mathcal{M}_{\Gamma}(g) + k_+(\bar{x}) + k_-(\bar{x}) \quad \text{in } \omega, \\ u_1 = 0 \quad \text{on } \partial\omega \end{cases}$$

and

$$\widehat{u}_1(\bar{x}, y) = -\sum_{j=1}^{2} \frac{\partial u_1}{\partial x_j}(\bar{x}) \chi_1^j(y) \quad \text{in } \omega \times Y_1.$$

Here, A^{hom} is the constant homogenized 2×2 matrix, defined, for $i, j \in \{1, 2\}$, by

$$A_{ij}^{\text{hom}} = \int_{Y_1} \left(a_{ij} - \sum_{k=1}^{3} a_{ik} \frac{\partial \chi_1^j}{\partial y_k} \right) \mathrm{d}y.$$

The function $\chi_1 = (\chi_1^1, \chi_1^2) \in (H^1_{\text{per}}(Y_1))^2$ is the weak solution of the cell problem ($j = 1, 2$):

$$\begin{cases} -\mathrm{d}iv_y(A(y)(\nabla_y \chi_1^j - e_j)) = 0 & \text{in } Y_1, \\ (A(y)(\nabla_y \chi_1^j - e_j)) \cdot n = 0 & \text{on } \Gamma, \\ (A(y)(\nabla_y \chi_1^j - e_j)) \cdot \nu_\pm = 0 & \text{on } \Sigma^1_\pm, \\ \mathcal{M}_\Gamma(\chi_1^j) = 0. \end{cases}$$

where n denotes the unit outward normal to Y_2 and $\nu_\pm = (0, 0, \pm 1)$.

• We remark that the limit problems satisfied by u_1 and u_2 can be written as a coupled system, consisting of a partial differential equation and an algebraic one and which is a modified stationary Barenblatt model:

$$\begin{cases} -\text{div}_{\bar{x}}\left(A^{\text{hom}}\nabla_{\bar{x}}u_1(\bar{x})\right) + |\Gamma|\mathcal{M}_\Gamma(h)(u_1 - u_2) = |Y_1|f(\bar{x}) + |\Gamma|\mathcal{M}_\Gamma(g) + k_+(\bar{x}) + k_-(\bar{x}) & \text{in } \omega, \\ -|\Gamma|\mathcal{M}_\Gamma(h)(u_1 - u_2) = |Y_2|f(\bar{x}) & \text{in } \omega, \\ u_1 = 0 & \text{on } \partial\omega \end{cases}$$

• The homogenized matrix A^{hom} is only of dimension two, but some information coming from the vertical direction of the microscopic problem is still present in it. The coefficients are influenced by the third local variable y_3, through the solution χ_1 of the cell problem.

• Conclusions. A diffusion problem in a thin composite material formed by two constituents separated by an imperfect interface was analyzed by using the periodic unfolding method, adapted to thin domains. The limit problem is described by a lower-dimensional modified Barenblatt system.

- At the limit, we obtain a lower-dimensional modified Barenblatt system (in which the dimension reduction phenomenon occurs). The effect of the small height of the domain Ω^ε is reflected in the fact that in the limit the diffusion occurs only in the horizontal bi-dimensional domain ω. The derivative with respect to the vertical variable x_3 does not appear in the equation, but, still, the solution of the effective problem keeps track of the local vertical variable y_3 via the values of the constant homogenized coefficients. The influence of the properly scaled flux jump is reflected in the appearance in the right-hand side of the homogenized equation of an additional source term, macroscopically distributed over ω.

THANK YOU FOR YOUR ATTENTION!